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ON THE RIEMANN ZETA FUNCTION

By P. R. TAYLOR (*Oxford*)

[Received 22 February 1944]

[Flight-Lieutenant P. R. Taylor was missing, believed killed, on active service in November 1943. The editors express their thanks to Mr. J. E. Rees for arranging the paper from the original manuscript and to Professor Titchmarsh for revising and completing the argument.]

Introduction

In this paper several disconnected results arising from researches on the Riemann zeta function are collected together.

In § 1 it is proved that $\xi_1(s+\frac{1}{2})-\xi_1(s-\frac{1}{2})$ has all its complex zeros on $\sigma = \frac{1}{2}$, where

$$\xi_1(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s).$$

In view of the equation

$$-\zeta(s) = \frac{s}{2}\zeta(s+1)+\frac{s(s+1)}{2.3}\zeta(s+2)+\ldots \quad (0<\sigma<1)$$

this may be a step towards the proof of the Riemann hypothesis, as the arguments of the zeta functions differ by unity in both cases.

To try to supplement this result, § 2 deals with geometrical properties of zeros of polynomials of the form

$$\sum_{r=0}^{n} a_r \binom{n}{r} z_n^{[r]},$$

where $z_n^{[r]} = (z-\frac{1}{2}n+\frac{1}{2})(z-\frac{1}{2}n+\frac{3}{2})\ldots(z-\frac{1}{2}n+r-\frac{1}{2})$.

The method employed is analogous to that of Pólya and Szegö,* and similar results are obtained.

In § 3 a formula is obtained for $\zeta(s)\zeta(s+\nu)$, where $0 \leqslant \nu < 1$, analogous to Riemann's formula for $\zeta(s)$

$$\tfrac{1}{2}\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \Re\left\{\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\int_L \frac{e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}x^{-s}\,dx\right\},$$

where $\Re(s) = \frac{1}{2}$ and L is a line inclined at $\frac{1}{4}\pi$ to the real axis and passing between 0 and 1. From the formula is derived the approximate functional equation for $\zeta(s)\zeta(s+\nu)$ on the line $\sigma = \frac{1}{2}(1-\nu)$; this includes a sharper form of the approximate functional equation for

* G. Pólya and G. Szegö, *Aufgaben und Lehrsätze* (Berlin, 1925), Abschn. 5, Kap. 2.

$\zeta^2(s)$ for $\sigma = \frac{1}{2}$ given by Hardy and Littlewood. A similar formula is stated for

$$\phi_k(s) = \sum \frac{a_{k,n}}{n^s},$$

where
$$\frac{1}{4k}\left\{\sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}\right\}^{2k} = \frac{1}{4k} + \sum_{n=1}^{\infty} a_{n,k} e^{-\pi n x}.$$

In § 4 a necessary and sufficient condition is given for the truth of Riemann's hypothesis in terms of $\log\left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi x}\right)$. No new ideas are involved.

In § 5 an elementary proof is given of van der Corput's lemma: *if $f'(x)$ is monotonic and $|f'(x)| < \frac{1}{2}$ for $a < x < b$, then*

$$\sum_{a<n<b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\, dx + O(1).$$

The extension by this method to the case where $\alpha < f'(x) < \beta$ is made, but not quite completely.

1. The zeros of a certain function

THEOREM. *All the complex zeros of*

$$F(s) = \xi_1(s+\tfrac{1}{2}) - \xi_1(s-\tfrac{1}{2}),$$

where
$$\xi_1(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

lie on the line $\sigma = \frac{1}{2}$.

Let
$$f(s) = \xi_1(s+\alpha) - \xi_1(s-\alpha),$$

where $\alpha > \frac{1}{2}$. Then $f(s)$ is regular except at simple poles

$$s = \alpha,\ 1+\alpha,\ -\alpha,\ 1-\alpha,$$

and satisfies the functional equation

$$f(s) = -f(1-s).$$

Since $f(1+\alpha+0) = -\infty$ and $f(\infty) = +\infty$, $f(s)$ has at least one zero in $(1+\alpha, \infty)$. Similarly it has at least one in $(-\infty, -\alpha)$, and also a zero at $s = \frac{1}{2}$.

There exists a number σ_0 such that $f(s)$ has no zeros when $\sigma \geqslant \sigma_0$. Let $N(T)$ be the number of zeros of $f(s)$ for $0 < t < T$; then

$$2N(T) - 1 + 2k = \frac{1}{2\pi}\Delta_R \operatorname{am} f(s),$$

where Δ_R denotes a variation round the rectangle $-T \leqslant t \leqslant T$,

$1-\sigma_0 \leqslant \sigma \leqslant \sigma_0$ and k is the number of zeros with $t = 0$, $\sigma > \frac{1}{2}$, unaccounted for. Hence

$$N(T) = \tfrac{1}{2} - k + \frac{1}{\pi}\Delta \operatorname{am} f(s),$$

where Δ means a variation from σ_0 to $\sigma_0 + iT$, then to $\frac{1}{2} + iT$, along straight lines. This follows from the functional equation for $f(s)$.

It follows that

$$N(T) = \tfrac{1}{2} - k + R(T) + S(T),$$

where $$R(T) = \frac{1}{\pi}\Delta \operatorname{am}[\pi^{-\frac{1}{2}s-\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\alpha)]$$

and $S(T) = \dfrac{1}{\pi}\Delta \operatorname{am}\{g(s)\}$, where

$$g(s) = \zeta(s+\alpha) - \pi^{\alpha}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\alpha)}\zeta(s-\alpha).$$

We now prove that

$$\int_0^T S(t)\,dt < A \log T. \tag{1}$$

Consider $\int \log g(s)\,ds$ round the rectangle

$$0 \leqslant t \leqslant T, \qquad \tfrac{1}{2} \leqslant \sigma \leqslant \beta.$$

It is equal to $$2\pi i \int_{\frac{1}{2}}^{\beta} \nu(\sigma, T)\,d\sigma,$$

$\nu(\sigma', T)$ being the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma > \sigma'$. Hence the real part vanishes.

Since $g(s) = 1 + O(e^{-A\sigma})$ as $\sigma \to \infty$ uniformly in t, the integral along $\sigma = \beta$ tends to 0 as $\beta \to \infty$, and hence

$$\int_0^T S(t)\,dt = \int_{\frac{1}{2}}^{\infty} \log|g(\sigma+iT)|\,d\sigma + K,$$

where K is the constant $-\int_{\frac{1}{2}}^{\infty} \log|g(\sigma)|\,d\sigma$. But $g(s) = O(t^A)$ uniformly in σ for $\sigma > \frac{1}{2}$. Hence

$$\int_0^T S(t)\,dt < A \log T.$$

Now let $N_0(T)$ be the number of zeros of $f(s)$ on $\sigma = \frac{1}{2}$ with

$0 < t < T$. They are given by $s = \frac{1}{2}+it$, where s satisfies the equation

$$\frac{\zeta(s+\alpha)}{\zeta(1-s+\alpha)} = \frac{\pi^{-\frac{1}{2}+\frac{1}{2}s-\frac{1}{2}\alpha}\Gamma(\frac{1}{2}-\frac{1}{2}s+\frac{1}{2}\alpha)}{\pi^{-\frac{1}{2}s-\frac{1}{2}\alpha}\Gamma(\frac{1}{2}s+\frac{1}{2}\alpha)},$$

and therefore by those values of s satisfying

$$\operatorname{am}\zeta(1-s+\alpha) = \operatorname{am}\{\pi^{-\frac{1}{2}s-\frac{1}{2}\alpha}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\alpha)\}-\nu\pi$$

for some integer value of ν, where both amplitudes are found by variation along $\sigma = \frac{1}{2}$.

But the amplitude on the right is the same as that found by variation from σ_0 to σ_0+it and then to $\frac{1}{2}+it$, where $s = \frac{1}{2}+it$, since the gamma function has no zeros or poles in the rectangle enclosed by these lines and $\sigma = \frac{1}{2}$ and $t = 0$. This amplitude is therefore equal to $\pi R(t)$.

$N_0(T)$ is equal to the number of points in which the lines $y = \nu$ ($\nu = 1, 2,\ldots$) cut the curve

$$y = R(x)+\frac{1}{\pi}\operatorname{am}\zeta(\tfrac{1}{2}+\alpha+ix)$$

for $0 < x < T$. This curve is ultimately monotonic, since the derivative of $\operatorname{am}\zeta(\frac{1}{2}+\alpha+ix)$ is bounded and that of $R(x)$ is asymptotic to $A\log x$, and it is easily seen from a graph that

$$N_0(T) \geqslant \left[R(T)+\frac{1}{\pi}\operatorname{am}\zeta(\tfrac{1}{2}+\alpha+iT)\right]$$

for all values of T. Hence

$$N_0(T) \geqslant R(T)+\frac{1}{\pi}\operatorname{am}\zeta(\tfrac{1}{2}+\alpha+iT)-1,$$

and consequently

$$N(T)-N_0(T) \leqslant \tfrac{3}{2}-k+S(T)-\frac{1}{\pi}\operatorname{am}\zeta(\tfrac{1}{2}+\alpha+iT).$$

Thus

$$\int_0^T \{N(t)-N_0(t)\}\,dt \leqslant (\tfrac{3}{2}-k)T+\int_0^T S(t)\,dt-\frac{1}{\pi}\int_0^T \operatorname{am}\zeta(\tfrac{1}{2}+\alpha+it)\,dt$$

$$\leqslant (\tfrac{3}{2}-k)T+A\log T$$

by (1) and the Dirichlet series for $\log\zeta(s)$. Since $N(t)-N_0(t)$ is non-decreasing,

$$\int_0^T \{N(t)-N_0(t)\}\,dt \geqslant (T-T')\{N(T')-N_0(T')\}$$

for $0 < T' < T$. Hence, dividing by T and making $T \to \infty$, we have

$$N(T') - N_0(T') \leqslant \tfrac{3}{2} - k.$$

Since $N(T')$ must exceed $N_0(T')$ by an even number, if at all, it follows that $N(T') = N_0(T')$.

The theorem follows by making $\alpha \to \frac{1}{2}$.

2. Some geometrical properties of the zeros of certain polynomials

Let $f(z) = \sum_{r=0}^{n} a_r \binom{n}{r} z_n^{[r]}$ be a polynomial of degree n ($z_n^{[r]}$ is defined in the introduction), and let the operation A_ζ be defined by

$$A_\zeta f(z) = (\zeta - z + \tfrac{1}{2}n) f(z + \tfrac{1}{2}) - (\zeta - z - \tfrac{1}{2}n) f(z - \tfrac{1}{2}).$$

Then $A_\zeta f(z)$ is a polynomial of degree $n-1$. We have

$$\begin{aligned} A_{\zeta_2}\{A_{\zeta_1} f(z)\} &= \{\zeta_2 - z + \tfrac{1}{2}(n-1)\}\{(\zeta_1 - z - \tfrac{1}{2} + \tfrac{1}{2}n) f(z+1) - (\zeta_1 - z - \tfrac{1}{2} - \tfrac{1}{2}n) f(z)\} - \\ &\quad - \{\zeta_2 - z - \tfrac{1}{2}(n-1)\}\{(\zeta_1 - z + \tfrac{1}{2} + \tfrac{1}{2}n) f(z) - (\zeta_1 - z + \tfrac{1}{2} - \tfrac{1}{2}n) f(z-1)\} \\ &= A_{\zeta_1}\{A_{\zeta_2} f(z)\}, \end{aligned}$$

so that the operation is commutative. We write

$$A_{\zeta_2}\{A_{\zeta_1} f(z)\} = A_{\zeta_1, \zeta_2} f(z).$$

Evidently $A_{\zeta_1, \ldots, \zeta_n}(z)$ will be a constant. We now calculate this in terms of the coefficients of the polynomial $g(z)$ whose zeros are $\zeta_1, \ldots, \zeta_n$. We have

$$A_\zeta f(z) = (\zeta - z)\Delta f(z) + n \nabla f(z),$$

where

$$\Delta z_n^{[r]} = (z - \tfrac{1}{2}n + 1) \ldots (z - \tfrac{1}{2}n + r) - (z - \tfrac{1}{2}n) \ldots (z - \tfrac{1}{2}n + r - 1) = r z_{n-1}^{[r-1]}$$

and

$$\nabla z_n^{[r]} = z_{n-1}^{[r-1]} (z - \tfrac{1}{2}n + \tfrac{1}{2}r) = z_{n-1}^{[r]} - \tfrac{1}{2} r z_{n-1}^{[r-1]}.$$

Therefore

$$\begin{aligned} A_\zeta f(z) &= \sum_{r=0}^{n} a_r \binom{n}{r} \{r z_{n-1}^{[r-1]} (\zeta - z) + n z_{n-1}^{[r]} - \tfrac{1}{2} n r z_{n-1}^{[r-1]}\} \\ &= \sum_{r=0}^{n} a_r \binom{n}{r} \{r z_{n-1}^{[r-1]} (\zeta - \tfrac{1}{2}n + r) - r z_{n-1}^{[r]} + n z_{n-1}^{[r]} - \tfrac{1}{2} n r z_{n-1}^{[r-1]}\} \\ &= n \sum_{r=0}^{n-1} \{(\zeta - n + r + 1) a_{r+1} + a_r\} \binom{n-1}{r} z_{n-1}^{[r]}. \end{aligned}$$

We shall say that $\sum_{r=0}^{n} a_r \binom{n}{r} z_n^{[r]}$ is *similar* to the infinite series

$$\sum_{k=1}^{\infty} \beta_k u(u-1)\ldots(u-k+1)+\alpha_0+\sum_{k=1}^{\infty} \frac{\alpha_k}{(u+1)(u+2)\ldots(u+k)} \qquad (A)$$

if $a_0 = \alpha_n, \quad a_1 = \alpha_{n-1}, \quad \ldots, \quad a_n = \alpha_0.$

Now

$$\left\{a_n+\frac{a_{n-1}}{u+1}+\frac{a_{n-2}}{(u+1)(u+2)}+\ldots+\frac{a_0}{(u+1)\ldots(u+n)}\right\}(u+\zeta)$$

$$= a_n u+a_n \zeta+\left\{a_{n-1}+\frac{a_{n-1}(\zeta-1)}{u+1}\right\}+\left\{\frac{a_{n-2}}{u+1}+\frac{a_{n-2}(\zeta-2)}{(u+1)(u+2)}\right\}+\ldots$$

$$= a_n u+a_n \zeta+a_{n-1}+\frac{a_{n-1}(\zeta-1)+a_{n-2}}{u+1}+\ldots,$$

which is similar to $\frac{1}{n} A_\zeta f(z).$

Hence $\frac{1}{n(n-1)} A_{\zeta_1,\zeta_2} f(z)$

is similar to

$$\left\{a_n+\frac{a_{n-1}}{u+1}+\ldots+\frac{a_0}{(u+1)\ldots(u+n)}\right\}(u+\zeta_1)(u+\zeta_2),$$

and finally $\frac{1}{n!} A_{\zeta_1,\ldots,\zeta_n} f(z)$

is similar to

$$(-1)^n\left\{a_n+\frac{a_{n-1}}{u+1}+\ldots+\frac{a_0}{(u+1)\ldots(u+n)}\right\}g(-u), \qquad (1)$$

where $g(u)$ is the polynomial whose roots are $\zeta_1,\ldots,\ \zeta_n$ and leading coefficient unity. Thus $\frac{1}{n!} A_{\zeta_1,\ldots,\zeta_n} f(z)$ must be the constant term in this expression when arranged in the form (A).

Suppose $g(u)$ expressed in the form

$$b_0+b_1 u+b_2 u(u-1)+\ldots+b_n u(u-1)\ldots(u-n+1).$$

Now (1) is the sum of terms like

$$\frac{u(u+1)\ldots(u+k-1)}{(u+1)\ldots(u+l)}.$$

If $k > l$, this is a polynomial with u as one factor, and therefore contributes nothing to the constant term in (1). If $k < l$, it can be

split up into partial fractions with no constant term, since the numerator is of lower order than the denominator. If $k = l$, we obviously get the constant term unity in its expression in the form (A). Therefore

$$\frac{(-1)^n}{n!} A_{\zeta_1,\ldots,\zeta_n} f(z) = a_n b_0 - a_{n-1} b_1 + \ldots + (-1)^n a_0 b_n.$$

We now show that the condition $A_{\zeta_1,\ldots,\zeta_n} f(z) = 0$ is equivalent to $A_{z_1,\ldots,z_n} g(z) = 0$, where $z_1, \ldots, z_n$ are the zeros of $f(z) = 0$. Suppose $g(z)$ expressed in the form

$$\sum_{r=0}^{n} B_r \binom{n}{r} z_n^{[r]}.$$

We have $$z_n^{[r]} = (z - \tfrac{1}{2}n + r - \tfrac{1}{2})^{[r]},$$

where $$\zeta^{[r]} = \zeta(\zeta-1)\ldots(\zeta-r+1).$$

Therefore

$$\sum_{r=0}^{n} B_r \binom{n}{r} z_n^{[r]} = \sum_{r=0}^{n} \sum_{\nu=0}^{r} B_r \binom{n}{r} \binom{r}{\nu} (-\tfrac{1}{2}n + \nu - \tfrac{1}{2})^{[r-\nu]} z^{[\nu]}$$

$$= \sum_{r=0}^{n} z^{[\nu]} \sum_{r=\nu}^{n} B_r \frac{\Gamma(n+1)\Gamma(r+1)}{\Gamma(r+1)\Gamma(n-r+1)\Gamma(\nu+1)\Gamma(r-\nu+1)} \frac{\Gamma(r-\tfrac{1}{2}n+\tfrac{1}{2})}{\Gamma(\nu-\tfrac{1}{2}n+\tfrac{1}{2})},$$

by Vandermonde's theorem. Thus the condition becomes

$$\sum_{\nu=0}^{n} (-1)^\nu a_{n-\nu} \sum_{r=\nu}^{n} B_r \frac{\Gamma(n+1)\Gamma(r-\tfrac{1}{2}n+\tfrac{1}{2})}{\Gamma(n-r+1)\Gamma(\nu+1)\Gamma(r-\nu+1)\Gamma(\nu-\tfrac{1}{2}n+\tfrac{1}{2})} = 0,$$

i.e.

$$\sin(\tfrac{1}{2}n+\tfrac{1}{2})\pi \sum_{r+s>n} a_s B_r \frac{\Gamma(r-\tfrac{1}{2}n+\tfrac{1}{2})\Gamma(s-\tfrac{1}{2}n+\tfrac{1}{2})}{\Gamma(n-r+1)\Gamma(n-s+1)\Gamma(r+s-n+1)},$$

which is unchanged by interchanging a and B.

Two polynomials related in this way will be called *apolar*. The geometrical properties of the zeros of two such polynomials are based on the following lemma.

LEMMA α. *If the zeros of $f(z)$ lie in the half-plane $\Re z > c$, and ζ lies outside this half-plane, then all the zeros of $A_\zeta f(z)$ lie in the half-plane.*

When z lies on the line $\Re z = c$, then

$$\frac{|\zeta - z + \tfrac{1}{2}n|}{|\zeta - z - \tfrac{1}{2}n|} < 1.$$

Also, if z_r is a zero of $f(z)$,

$$\left|\frac{z-z_r+\frac{1}{2}}{z-z_r-\frac{1}{2}}\right| < 1.$$

Therefore
$$\left|\frac{\{\zeta-z+\frac{1}{2}n\}f(z+\frac{1}{2})}{\{\zeta-z-\frac{1}{2}n\}f(z-\frac{1}{2})}\right| < 1$$

on the line $\Re(z) = c$. Hence the amplitude of the function

$$F(z) = \frac{\{\zeta-z+\frac{1}{2}n\}f(z+\frac{1}{2})}{\{\zeta-z-\frac{1}{2}n\}f(z-\frac{1}{2})} - 1$$

cannot change by more than π as z moves along this line. But

$$F(z) = \frac{A}{z^2} + O\left(\frac{1}{z^3}\right) \quad \text{as} \quad |z| \to \infty.$$

Therefore the change in am $F(z)$, as z moves along the arc of a large semicircle to the left of $\Re(z) = c$, tends to 2π. This change just accounts for the only pole $z = \zeta - \frac{1}{2}n$ of $F(z)$ inside the contour formed by the large semicircle and $\Re(z) = c$, and there is not sufficient change left for a zero to occur. Hence all the zeros of $F(z)$, and therefore of $A_\zeta f(z)$, lie in $\Re(z) > c$.

LEMMA β. *If $A_\zeta f(z) = 0$ identically, then ζ is the mid-point of two zeros of $f(z)$.*

We have

$$(\zeta-z+\tfrac{1}{2}n)f(z+\tfrac{1}{2}) = (\zeta-z-\tfrac{1}{2}n)f(z-\tfrac{1}{2}).$$

Now
$$\frac{\zeta-z+\frac{1}{2}n}{\zeta-z-\frac{1}{2}n} = \frac{g(z-\frac{1}{2})}{g(z+\frac{1}{2})}$$

where

$$g(z) = (\zeta-z-\tfrac{1}{2}n+\tfrac{1}{2})(\zeta-z-\tfrac{1}{2}n+\tfrac{3}{2})\ldots(\zeta-z+\tfrac{1}{2}n-\tfrac{1}{2}).$$

Therefore $f(z)/g(z)$ is a rational function of period unity and is accordingly a constant.

Thus $f(z) = Kg(z)$, whose zeros lie on a line with ζ midway between the first and last.

THEOREM. *If $f(z)$ and $g(z)$ are apolar, then any half-plane $\Re(z) > c$ containing all the zeros of $f(z)$ contains at least one zero of $g(z)$.*

Let $\zeta_1, \ldots, \zeta_n$ be the zeros of $g(z)$, and let all the zeros of $f(z)$ lie in a half-plane H. Then, if ζ_1 does not lie in H, every zero of $A_{\zeta_1} f(z)$ lies in H, by Lemma α. If f and g are apolar, there must be a smallest

k for which $A_{\zeta_1,\ldots,\zeta_k}f(z) = 0$ identically. But, if $\zeta_1,\ldots,\zeta_{k-1}$ all lie outside H, all the zeros of $A_{\zeta_1,\ldots,\zeta_{k-1}}f(z)$ lie in H, and therefore, by Lemma β, ζ_k also lies in H. Hence at least one zero lies inside H.

3. A formula for $\zeta(s)\zeta(s+\nu)$

From the functional equation for the zeta function

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}\pi s\,\Gamma(s)\zeta(s)$$

we derive the equation

$$\zeta(1-s)\zeta(1-s-\nu) = 2^{2-2s-\nu}\pi^{-2s-\nu}\cos\tfrac{1}{2}\pi s\cos\tfrac{1}{2}\pi(s+\nu)\Gamma(s)\Gamma(s+\nu)\zeta(s)\zeta(s+\nu). \quad (1)$$

We suppose $0 < \nu < 1$. Then the function

$$Z(2s) = (2\pi)^{-s}\cos\tfrac{1}{2}\pi s\,\Gamma(s)\zeta(s)\zeta(s+\nu)$$

satisfies the equation

$$Z(2s) = Z\{2(1-\nu)-2s\}, \quad (1')$$

and $Z(s)$ is real on the line $\sigma = 1-\nu$.

The equation about to be proved will give an approximate formula for $Z(s)$ on this line. The result could easily be extended to other regions. Put

$$\zeta(s)\zeta(s+\nu) = \sum \frac{a_n}{n^s}.$$

Since

$$x^\nu K_\nu(x) = \frac{1}{2\pi i}\int_{(c)} 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)x^{-s}\,ds \quad (c > 0),$$

we have

$$\sum a_n(x\sqrt{n})^\nu K_\nu(x\sqrt{n}) = \frac{1}{2\pi i}\int_{(c)} 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)\zeta(\tfrac{1}{2}s)\zeta(\tfrac{1}{2}s+\nu)x^{-s}\,ds \quad (2 < c < 3).$$

Move the contour to

$$\sigma = c' = 2(1-\nu)-c.$$

The poles passed over give a residue of the form

$$R(x) = Ax^{-2}+Bx^{-2+2\nu}+C+Dx^{2\nu}$$

and the remaining integral is

$$\frac{1}{2\pi i}\int_{(c')} \frac{2^{-4+2s+2\nu}\pi^{s+\nu}\zeta(1-\frac{1}{2}s)\zeta(1-\nu-\frac{1}{2}s)}{\cos\frac{1}{4}\pi s\cos(\frac{1}{4}\pi s+\frac{1}{2}\pi\nu)}x^{-s}\,ds = \frac{1}{2\pi i}\int_{(c)} \frac{2^{-2\nu-2s}\pi^{2-\nu-s}\zeta(\frac{1}{2}s)\zeta(\frac{1}{2}s+\nu)}{\sin\frac{1}{4}\pi s\sin(\frac{1}{4}\pi s+\frac{1}{2}\pi\nu)}x^{s-2(1-\nu)}\,ds \quad \text{by (1).}$$

The formula

$$\frac{1}{2\pi i}\int_{(c)} \frac{y^{-s}}{\sin\frac{1}{4}\pi s\sin(\frac{1}{4}\pi s+\frac{1}{2}\pi\nu)}\,ds = \frac{1}{\pi\sin\frac{1}{2}\pi\nu}\left(\frac{1-y^{2\nu}}{1-y^4}\right)$$

is easily found by shifting the line of integration to $\sigma = c-4$ and evaluating the residues. Using this we get

$$\sum a_n(x\sqrt{n})^\nu K_\nu(x\sqrt{n}) = R(x)+\frac{2^{-2\nu}\pi^{1-\nu}x^2}{\sin\frac{1}{2}\pi\nu}\sum a_n\frac{x^{2\nu}-(4\pi\sqrt{n})^{2\nu}}{x^4-(4\pi\sqrt{n})^4}. \quad (2)$$

Hence the function $\phi(x)$, defined for $\Re(x) > 0$ to be

$$\sum a_n n^{\frac{1}{2}\nu}K_\nu(x\sqrt{n}),$$

can be continued throughout the whole plane. It has a branch-point at the origin but is one-valued in the plane slit along the negative real axis, with simple poles at the points $x = \pm 4\pi i\sqrt{n}$, the residue at $+4\pi i\sqrt{n}$ being proportional to $a_n n^{\frac{1}{2}\nu-\frac{1}{4}}$.

This function $\phi(x)$ satisfies the inequalities

$$\phi(x) = O(e^{-A\Re(x)}) \quad (\Re(x) > A) \quad (3)$$

and

$$\phi(x) = O(|x|^A) \quad (4)$$

in the whole plane excluding the negative real axis by a strip of breadth unity and the points $\pm 4\pi i\sqrt{n}$ by circles with centres at these points and radii $1/\sqrt{n}$. The inequality (3) easily follows from the formula

$$K_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-z} \quad \text{and} \quad a_n = O(n^\epsilon). \quad (5)$$

To prove (4), divide up the sum

$$\sum a_n\frac{x^{2\nu}-(4\pi\sqrt{n})^{2\nu}}{x^4-(4\pi\sqrt{n})^4}$$

into two parts: those $O(|x|^A)$ terms for which $|x^4-(4\pi\sqrt{n})^4| > A/\sqrt{n}$ and the remainder for which $|x^4-(4\pi\sqrt{n})^4| > An^2$. Obviously each part is $O(|x|^A)$. The argument does not apply to the neighbourhood of the points $4\pi\sqrt{n}$, but this is clearly covered by (3). Now consider

$$\int_L e^{\pi i x^2}K_\nu(2\pi i a x)\phi(2\pi i x\sqrt{2})x\,dx,$$

L being a line passing between 0 and $\sqrt{2}$ inclined at $\frac{1}{4}\pi$ to the real axis. If $\Im(a) < -1$, it follows from (4), (5) that L may be bent down over the poles of $\phi(2\pi i x\sqrt{2})$ on the real axis as far as the line $\Im(z) = -1$. The residues at the poles total

$$C\sum a_n n^{\frac{1}{2}\nu}K_\nu\{2\pi i a\sqrt{(2n)}\},$$

where C is a constant, and the remaining integral is now

$$\sum a_n n^{\frac{1}{2}\nu} \int_L e^{\pi i x^2} K_\nu(2\pi i a x) K_\nu\{2\pi i x\sqrt{(2n)}\} x\, dx$$

since on the line of integration $\phi(2\pi i x\sqrt{2})$ is now represented by the absolutely convergent series

$$\sum a_n n^{\frac{1}{2}\nu} K_\nu\{2\pi i x\sqrt{(2n)}\}.$$

To evaluate the integral

$$\int_L e^{\pi i x^2} K_\nu(2\pi i a x) K_\nu(2\pi i b x) x\, dx$$

we use the formula*

$$K_\nu(Z)K_\nu(z) = \tfrac{1}{2}\int_0^\infty \exp\left[-\frac{v}{2} - \frac{Z^2+z^2}{2v}\right] K_\nu\left(\frac{Zz}{v}\right)\frac{dv}{v}.$$

We put $\frac{1}{2}Zz = s$, $v = 2sx$, keep $(Z^2+z^2)/Zz$ fixed, and regard this as a formula connecting two Laplace transforms. The reciprocal formula, after a suitable change of the variable of integration, is found to be

$$\int_L e^{\pi i x^2} K_\nu(2\pi i a x) K_\nu(2\pi i b x) x\, dx = \tfrac{1}{2} e^{-\pi i(a^2+b^2)} K_\nu(2\pi i a b).$$

Putting in this value for the integral, we find

$$\int_L e^{\pi i x^2} K_\nu(2\pi i a x)\phi_\nu(2\pi i x\sqrt{2}) x\, dx = \tfrac{1}{2}(e^{-\pi i a^2} + C)\phi(2\pi i a\sqrt{2}),$$

where $C = -1$ since the left-hand side has no pole at $a = \sqrt{2}$.

Now multiply both sides by $a^{s+\nu-1}$ and integrate from 0 to $\infty e^{-\frac{1}{4}i\pi}$. We have

$$\int_0^\infty K_\nu(2\pi i a x) a^{s+\nu-1}\, da = (2\pi i x)^{-s-\nu}\int_0^\infty K_\nu(a) a^{s+\nu-1}\, da$$

$$= (2\pi i x)^{-s-\nu} 2^{s+\nu-2}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu).$$

Hence

$$2^{s+\nu-1}(2\pi i)^{-s-\nu}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)\int_L e^{\pi i x^2}\phi(2\pi i x\sqrt{2}) x^{1-s-\nu}\, dx$$

$$= -2^{s+\nu-2}(2\pi i\sqrt{2})^{-s-\nu}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}s+\nu)\zeta(\tfrac{1}{2}s)\zeta(\tfrac{1}{2}s+\nu) +$$

$$+ \int_0^{\infty e^{-\frac{1}{4}i\pi}} e^{-\pi i a^2}\phi(2\pi i a\sqrt{2}) a^{s+\nu-1}\, da. \quad (6)$$

* G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), 439.

Write the last integral as

$$e^{-\pi i(s+\nu)} \int_0^{\infty e^{\frac{1}{4}\pi i}} e^{-\pi i a^2} \phi(-2\pi i a\sqrt{2}) a^{s+\nu-1}\, da.$$

It is now easy to verify that $\phi(2\pi y\sqrt{2})$ in this formula may be replaced by $g(y)$, where

$$g(y) = \phi(2\pi y\sqrt{2}) - \{C(2\pi y\sqrt{2})^{-\nu} + D(2\pi y\sqrt{2})^{\nu}\}.$$

The extra terms depend merely on gamma functions.

We now express

$$\int_0^{\infty e^{\frac{1}{4}\pi i}} e^{-\pi i a^2} g(-ia) a^{s+\nu-1}\, da$$

in terms of

$$I = \int_{\bar{L}} e^{-\pi i a^2} g(-ia) a^{s+\nu-1}\, da \quad \text{and} \quad J = \int_{\bar{L}} e^{-\pi i a^2} g(a) a^{s+\nu-1}\, da,$$

$\bar{L}$ being the reflection of L in the real axis. We have

$$g(z) = pz^{-2-\nu} + qz^{-2+\nu} + rz^{2-\nu} \sum a_n \frac{z^{2\nu} - (2n)^{\nu}}{z^4 - (2n)^2},$$

where p, q, r are constants. Write

$$g_1(z) = pz^{-2-\nu} - rz^{2-\nu} \sum a_n \frac{(2n)^{\nu}}{z^4 - (2n)^2},$$

$$g_2(z) = qz^{-2+\nu} + rz^{2+\nu} \sum a_n \frac{1}{z^4 - (2n)^2}.$$

Then

$$\int_{\bar{L}} g_1(a) = (e^{-\pi i s} - 1) \int_0^{\infty e^{\frac{1}{4}\pi i}} g_1(a), \qquad \int_{\bar{L}} g_2(a) = (e^{-\pi i(s+2\nu)} - 1) \int_0^{\infty e^{\frac{1}{4}\pi i}} g_2(a),$$

where $\int_{\bar{L}} g_1(a) = \int_{\bar{L}} e^{-\pi i a^2} g_1(a) a^{s+\nu-1}\, da$, and so on.

Also

$$\left.\begin{aligned} g_1(a) &= -e^{-\frac{1}{2}\pi i\nu} g_1(-ia) \\ g_2(a) &= -e^{\frac{1}{2}\pi i\nu} g_2(-ia) \\ g(-ia) &= g_1(-ia) + g_2(-ia) \\ g(a) &= -e^{-\frac{1}{2}\pi i\nu} g_1(-ia) - e^{\frac{1}{2}\pi i\nu} g_2(-ia) \end{aligned}\right\}. \tag{7}$$

Accordingly,

$$g_1(-ia) = \frac{e^{\frac{1}{2}\pi i\nu} g(-ia) + g(a)}{e^{\frac{1}{2}\pi i\nu} - e^{-\frac{1}{2}\pi i\nu}}, \qquad g_2(-ia) = \frac{e^{-\frac{1}{2}\pi i\nu} g(-ia) + g(a)}{e^{-\frac{1}{2}\pi i\nu} - e^{\frac{1}{2}\pi i\nu}}.$$

Hence

$$\int_0^{\infty e^{\frac{1}{4}\pi i}} g(-ia) = \int g_1 + \int g_2$$

$$= -\frac{1}{(e^{-\pi i s}-1)(e^{-\pi i(s+2\nu)}-1)}\{(e^{-\pi i(s+\nu)}+1)I + e^{-\pi i(s+\nu)}(e^{\frac{1}{2}\pi i\nu}+e^{-\frac{1}{2}\pi i\nu})J\}.$$

On substituting this in (6) and dividing by a suitable factor and reducing, we get

$$\tfrac{1}{2}Z(s) = -2^{\frac{1}{2}\nu}\pi^{-\frac{1}{2}s}\cos\tfrac{1}{4}\pi s\,\Gamma(\tfrac{1}{2}s)\bar{I} - \\ -\frac{2^{\frac{1}{2}\nu}\pi^{\frac{1}{2}s+\nu-1}\Gamma(1-\nu-\frac{1}{2}s)}{\sin\frac{1}{4}\pi s}[I\cos\tfrac{1}{2}\pi(s+\nu)+Je^{-\frac{1}{2}\pi i(s+\nu)}\cos\tfrac{1}{2}\pi\nu],$$

where $\bar{I}$ is the conjugate of I when $\sigma = 1-\nu$.

Now suppose $\sigma = 1-\nu$. Then $Z(s)$ is real, and the formula is unaltered by replacing each item by its conjugate and adding. This gives

$$\tfrac{1}{2}Z(s) = -\Re[2^{\frac{1}{2}\nu}\pi^{\frac{1}{2}s+\nu-1}\Gamma(1-\nu-\tfrac{1}{2}s)\operatorname{cosec}\tfrac{1}{4}\pi s\times \\ \times\{I\cos\tfrac{1}{2}\pi(s+\nu)+(I+Je^{-\frac{1}{2}\pi i(s+\nu)})\cos\tfrac{1}{2}\pi\nu\}]. \quad (8)$$

Another formula of the same type might be worth noticing. Let

$$\frac{1}{4k}\Big(\sum_{n=-\infty}^{\infty} e^{-n^2\pi x}\Big)^{2k} = \frac{1}{4k} + \sum_{n=1}^{\infty} a_{n,k}\,e^{-n\pi x}.$$

Then the function $\phi_k(s) = \sum \dfrac{a_{n,k}}{n^s}$

satisfies the equation

$$\pi^{-s}\Gamma(s)\phi_k(s) = \pi^{-k+s}\Gamma(k-s)\phi_k(k-s).$$

If we put $\psi_k(x) = \dfrac{1}{2k} + 2x^2\sum_{n=1}^{\infty}\dfrac{(-1)^n a_{n,k}}{x^2-n},$

then, when $\Im(x) < 0$, we have

$$\psi_k(x) = \pi(4ix)^{k+1}\sum_{n=0}^{\infty} b_n\{\sqrt{(2n+\tfrac{1}{2}k)}\}^{1-k}K_{1-k}\{2\pi ix\sqrt{(2n+\tfrac{1}{2}k)}\},$$

where b_n is independent of x, and

$$\pi^{-s}\Gamma(s)\phi_k(x) = -\frac{\pi^{-s}\Gamma(s)}{2\pi i}\int_L e^{\pi ix^2}\psi_k(x)x^{-2s-1}\,dx + \\ +\frac{\pi^{-k+s}\Gamma(k-s)}{2\pi i}\int_{\bar{L}} e^{-\pi ix^2}\psi_k(x)x^{2s-2k-1}\,dx.$$

If $k = \frac{1}{2}$, this reduces to Riemann's formula for $\zeta(s)$

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\int_{\bar{L}} \frac{e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}} x^{-s}\,dx - \\ -\pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\int_{\bar{L}} \frac{e^{-\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}} x^{s-1}\,dx.$$

We next approximate to I and J. First we show that the term $Je^{-\frac{1}{2}\pi i(s+v)}$ is negligible. Consider

$$J = \int_{\bar{L}} e^{-\pi i x^2} g(x) x^{s+v-1}\,dx.$$

The saddle-point of $e^{-\pi i x^2}x^{s+v-1}$ is at $x = \eta$, where $2\pi i\eta^2 = s+v-1$. Suppose that $s = 1-v+it$, where $t > 0$, so that $\eta = +\sqrt{(t/2\pi)}$ is real and positive. Then

$$Je^{-\frac{1}{2}\pi i(s+v)} = \sum_{n<t/4\pi} a_n n^{\frac{1}{2}s+v-1} + \int_{\bar{L}'} e^{-\pi i x^2} g(x) x^{s+v-1}\,dx,$$

$\bar{L}'$ being the line through η parallel to $\bar{L}$. Write

$$g(x) = \phi(2\pi x\sqrt{2}) - \{C(2\pi x\sqrt{2})^{-v} + D(2\pi x\sqrt{2})^{v}\}.$$

The bracketed terms give rise to gamma functions. If these are taken together with their conjugates in (8), it is found that they contribute only $O(e^{-At})$ to the integral. To discuss the term involving $\phi(x)$, we follow Siegel,* and write it as

$$\eta^{s+v-1}e^{-\pi i\eta^2}\left\{\int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}\phi(2\pi x\sqrt{2})\,dx + \int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}\phi(2\pi x\sqrt{2})\omega(z)\,dx\right\},$$

where $z = x-\eta$ and

$$\omega(z) = \exp\left[(s+v-1)\left\{\log\left(1+\frac{z}{\eta}\right)-\frac{z}{\eta}+\frac{1}{2}\frac{z^2}{\eta^2}\right\}\right]-1,$$

which satisfies the inequalities

$$|e^{-2\pi i z^2}\omega(z)| \leqslant \begin{cases} Ae^{-\frac{1}{2}\pi|z|^2} & (|z| > \frac{1}{2}\eta), \\ Ae^{-2\pi|z|^2}\left\{\exp\left(\frac{4\pi}{3}\left|\frac{z}{\eta}\right|^3\right)-1\right\} & (|z| \leqslant \frac{1}{2}\eta) \end{cases} \tag{9}$$

* Siegel, 'Über Riemann's Nachlass zur Analytischen Zahlentheorie': *Quellen und Studien zur Geschichte d. Math.* Abt. B, Bd. 2.

if x lies on $\bar{L}'$. Using (3) and (4) we find that

$$\int_{\bar{L}'} e^{-2\pi i z^2}\omega(z)\phi(2\pi x\sqrt{2})\,dx = O\Big\{e^{-A\eta}\int_{|z|\leqslant\frac{1}{2}\eta} e^{-2\pi|z|^2}(e^{\frac{1}{2}\pi|z/\eta|^2}-1)e^{-Az}\,dz\Big\}+$$

$$+O\Big\{\int_{|z|>\frac{1}{2}\eta} e^{-\frac{1}{2}\pi|z|^2}|z+\eta|^A\,dz\Big\} = O(e^{-A\eta}).$$

Also

$$\int_{\bar{L}'} e^{-2\pi i z^2}\phi(2\pi x\sqrt{2})\,dx$$

$$= O\Big\{e^{-A\eta}\int_{|z|\leqslant\frac{1}{2}\eta} e^{-2\pi|z|^2-Az}\,dz\Big\}+O\Big\{\int_{|z|>\frac{1}{2}\eta} e^{-2\pi|z|^2}|z+\eta|^A\,dz\Big\} = O(e^{-A\eta}).$$

This shows that $$Je^{-\frac{1}{2}\pi i(s+\nu)} \tag{10}$$

is negligible in comparison with $I\cos\frac{1}{2}\pi(s+\nu)$.

We now perform the same operation on I as on J and write

$$I = \sum_{\eta<|l|\leqslant 4\pi} a_n n^{\frac{1}{2}s+\nu-1}+\eta^{s+\nu-1}e^{-\pi i\eta^2}\times$$

$$\times\Big\{\int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}g(-ix)\,dx+\int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}g(-ix)\omega(z)\,dx\Big\}. \tag{11}$$

Now we have, if $x=\alpha+i\beta$ and $\beta>0$,

$$|g(-ix)| = O(|x|^\nu)+O\{\textstyle\sum a_n n^{\frac{1}{2}\nu}K_\nu(2\pi\beta\sqrt{(2n)})\}$$

since $$|K_\nu(x+iy)|\leqslant K_\nu(x),$$

i.e. $$g(-ix) = O(|x|^\nu)+O(|\beta|^{-2-\nu}).$$

This inequality holds also when $\beta<0$, since, from (7),

$$g(-ix)+2g(x)\cos\tfrac{1}{2}\pi\nu+g(ix) = 0. \tag{12}$$

Hence, using (9),

$$\int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}g(-ix)\omega(z)\,dx = O\Big\{\int_{|z|\leqslant\frac{1}{2}\eta} e^{-2\eta|z|^2+\frac{1}{2}\pi|z/\eta|^3}\Big|\frac{z}{\eta}\Big|^3\{\eta^\nu+|z|^{-2-\nu}\}\,dx\Big\}+$$

$$+O\Big\{\int_{|z|>\frac{1}{2}\eta} e^{-\frac{1}{2}\pi|z|^2}|x|^\nu\,dx\Big\} = O(\eta^{-1+\nu}).$$

It remains to consider

$$\int_{\bar{L}'} e^{-2\pi i(x-\eta)^2}g(-ix)\,dx.$$

Suppose

$$g(x) = C_1x^{-\nu}+D_1x^\nu+\textstyle\sum a_n n^{\frac{1}{2}\nu}K_\nu\{2\pi x\sqrt{(2n)}\}\quad(\Re(x)>0).$$

Then using (12) we may write the integral as

$$\sum a_n n^{\frac{1}{2}\nu} \int_{\Im x>0} e^{-2\pi i(x-\eta)^2} K_\nu\{-2\pi i x\sqrt{(2n)}\}\, dx -$$

$$- \sum a_n n^{\frac{1}{2}\nu} \int_{\Im x<0} e^{-2\pi i(x-\eta)^2} \{2K_\nu[2\pi x\sqrt{(2n)}]\cos\tfrac{1}{2}\pi\nu + K_\nu[2\pi i x\sqrt{(2n)}]\}\, dx +$$

$$+ \int_{\Im x>0} e^{-2\pi i(x-\eta)^2} \{C_1(-ix)^{-\nu} + D_1(-ix)^\nu\}\, dx -$$

$$- \int_{\Im x<0} e^{-2\pi i(x-\eta)^2} \{2(C_1 x^{-\nu} + D_1 x^\nu)\cos\tfrac{1}{2}\pi\nu + C_1(ix)^{-\nu} + D_1(ix)^\nu\}\, dx$$

$$= I_1 + I_2 + I_3 + I_4,$$

$$I_3 + I_4 = e^{\frac{1}{2}\pi i} \int_0^\infty e^{-2\pi r^2} [\{C_1(-i\eta + re^{\frac{1}{2}\pi i})^{-\nu} + D_1(-i\eta + re^{\frac{1}{2}\pi i})^\nu\} -$$

$$- 2\{C_1(\eta + re^{-\frac{1}{2}i\pi})^{-\nu} + D_1(\eta + re^{-\frac{1}{2}i\pi})^\nu\}\cos\tfrac{1}{2}\pi\nu -$$

$$- \{C_1(i\eta + re^{\frac{1}{2}\pi i})^{-\nu} + D_1(i\eta + re^{\frac{1}{2}\pi i})^\nu\}]\, dr.$$

Now

$$\int_0^\infty e^{-2\pi r^2}(-i\eta + re^{\frac{1}{2}\pi i})^\nu\, dr = e^{-\frac{1}{2}\pi i\nu}\eta^\nu \int_0^\infty e^{-2\pi r^2}\left\{1 + \frac{\nu r e^{\frac{1}{2}\pi i}}{\eta} + O\left(\frac{r^2}{\eta^2}\right)\right\} dr$$

$$= e^{-\frac{1}{2}\pi i\nu}\eta^\nu\left\{\frac{1}{2\sqrt{2}} + \frac{\nu e^{\frac{1}{2}\pi i}}{4\pi\eta} + O\left(\frac{1}{\eta^2}\right)\right\},$$

and similarly for the other terms. This gives

$$I_3 + I_4 = \frac{e^{\frac{1}{2}\pi i}}{\sqrt{2}}\{C_1\eta^{-\nu}e^{-\frac{1}{2}\pi i\nu} + D_1\eta^\nu e^{\frac{1}{2}\pi i\nu}\} + O\left(\frac{1}{\eta}\right). \tag{13}$$

Also

$$I_1 + I_2 = e^{\frac{1}{2}\pi i}\sum a_n n^{\frac{1}{2}\nu} \int_0^\infty e^{-2\pi r^2}[K_\nu\{2\pi\sqrt{(2n)}(-i\eta + re^{\frac{1}{2}i\pi})\} -$$

$$- 2K_\nu\{2\pi\sqrt{(2n)}(\eta + re^{-\frac{1}{2}\pi i})\}\cos\tfrac{1}{2}\pi\nu - K_\nu\{2\pi\sqrt{(2n)}(i\eta + re^{\frac{1}{2}\pi i})\}]\, dr.$$

Since $K_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-x} + O\left\{\left|\frac{e^{-x}}{x^{\frac{3}{2}}}\right|\right\}$, each K-function may be replaced by its dominant term with error

$$O(\eta^{-\frac{3}{2}})\sum a_n n^{\frac{1}{2}\nu} n^{-\frac{1}{4}},$$

since

$$\int_0^\infty e^{-x^2 - 2nx}\, dx = \frac{1}{n}\int_0^\infty e^{-(x^2/n^2) - 2x}\, dx \sim \frac{1}{2n}.$$

The series is convergent if $\nu < \frac{1}{2}$. We assume this to be true.

Then

$$I_1+I_2=\frac{1}{\pi}e^{\frac{1}{4}\pi i}\sum a_n n^{\frac{1}{4}\nu}\int_0^\infty e^{-2\pi r^2}\times$$

$$\times\left[\frac{e^{2\pi i\eta\sqrt{(2n)}-2\pi re^{\frac{1}{4}\pi i}\sqrt{(2n)}}}{(2n)^{\frac{1}{4}}(-i\eta+re^{\frac{1}{4}\pi i})^{\frac{1}{2}}}-2\cos\tfrac{1}{2}\pi\nu\frac{e^{-2\pi i\eta\sqrt{(2n)}-2\pi re^{-\frac{1}{4}\pi i}\sqrt{(2n)}}}{(2n)^{\frac{1}{4}}(\eta+re^{-\frac{1}{4}\pi i})^{\frac{1}{2}}}-\right.$$

$$\left.-\frac{e^{-2\pi i\eta\sqrt{(2n)}-2\pi re^{\frac{1}{4}\pi i}\sqrt{(2n)}}}{(2n)^{\frac{1}{4}}(i\eta+re^{\frac{1}{4}\pi i})^{\frac{1}{2}}}+O(\eta^{-\frac{1}{2}})\right]dr.$$

To approximate to this integral we write

$$(-i\eta+re^{\frac{1}{4}\pi i})^{-\frac{1}{2}}=\eta^{-\frac{1}{2}}e^{\frac{1}{4}\pi i}\left\{1+O\left(\frac{r}{\eta}\right)\right\}$$

and so forth, and then

$$\int_0^\infty e^{-2\pi r^2-2\pi re^{\frac{1}{4}\pi i}\sqrt{(2n)}}dr=\frac{1}{2\pi e^{\frac{1}{4}\pi i}\sqrt{(2n)}}\int_0^\infty e^{ir^2/4\pi n-r}\,dr$$

$$=\frac{1}{2\pi e^{\frac{1}{4}\pi i}\sqrt{(2n)}}+O\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

This gives finally

$$I_1+I_2=\frac{e^{\frac{1}{4}\pi i}}{\pi\eta^{\frac{1}{2}}}\sum\frac{a_n n^{\frac{1}{4}\nu}}{(2n)^{\frac{1}{4}}}\left[\frac{e^{2\pi i\eta\sqrt{(2n)}}}{2\pi\sqrt{(2n)}}+i\frac{e^{-2\pi i\eta\sqrt{(2n)}}}{2\pi\sqrt{(2n)}}\right]+O(\eta^{-\frac{1}{2}})$$

$$=-\frac{1}{2^{\frac{1}{2}}\pi^2\eta^{\frac{1}{2}}}\sum a_n n^{\frac{1}{2}\nu-\frac{1}{4}}\cos\{2\pi\eta\sqrt{(2n)}-\tfrac{1}{4}\pi\}+O(\eta^{-\frac{1}{2}}). \quad (14)$$

Now put $2^{1-\frac{1}{2}s}\pi^{-\frac{1}{2}s}\cos\frac{1}{4}\pi s\,\Gamma(\frac{1}{2}s)=Re^{i\vartheta}$ when $s=1-\nu+it$. Collecting up the results (10), (11), (13), (14), (8) and using Stirling's asymptotic formula for the gamma function, we get finally

$$\left(\frac{t}{4\pi}\right)^{\frac{1}{2}\nu}|\zeta\{\tfrac{1}{2}(1+\nu)+\tfrac{1}{2}it\}|^2$$

$$=2\sum_{n<t/4\pi}\frac{a_n\cos(\vartheta-\frac{1}{2}t\log n)}{n^{\frac{1}{2}-\frac{1}{2}\nu}}-4\{C(4\pi t)^{-\frac{1}{2}\nu}+D(4\pi t)^{\frac{1}{2}\nu}\}\cos\tfrac{1}{2}\pi\nu+$$

$$+\frac{4\pi^{-\frac{1}{2}}t^{-\frac{1}{4}}}{2^{\frac{1}{4}}}\sum a_n n^{\frac{1}{2}\nu-\frac{1}{4}}\cos\{\sqrt{(4\pi tn)}-\tfrac{1}{4}\pi\}+O(t^{-\frac{1}{4}})\quad(0<\nu<\tfrac{1}{2}),$$

where we find $C=-2^{\nu-2}\Gamma(\nu)\zeta(\nu)$, $D=-2^{-\nu-2}\Gamma(-\nu)\zeta(-\nu)$.

This formula also holds when $\nu=0$ if we interpret

$$C(4\pi t)^{-\frac{1}{2}\nu}+D(4\pi t)^{\frac{1}{2}\nu}$$

as its limit for fixed t when $\nu\to 0$. To prove this it is only necessary

Conversely, suppose the Riemann hypothesis true. We have

$$\pi^{-s}f(s)\Gamma(s) = \int_0^\infty \phi(x)x^{s-1}\,dx \quad (\Re(s) > 1).$$

Therefore

$$\phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \pi^{-s}f(s)\Gamma(s)x^{-s}\,ds \quad (c > 1).$$

As $|t| \to \infty$, $f(s) = O(|t|^A)$ for any fixed $\sigma > -\frac{3}{4}$. Therefore, since $\Gamma(s) = O(e^{-\frac{1}{2}\pi|t|}|t|^A)$, $f(s)\Gamma(s) \to 0$ along any line $\sigma =$ constant. It is thus permissible for the line of integration to be moved from $\sigma = c$ to $\sigma = -\frac{3}{4}+\epsilon$, if residue terms at $s = 1$ and $s = 0$ are added in. The integral along $\sigma = -\frac{3}{4}+\epsilon$ is obviously $O(x^{\frac{3}{4}-\epsilon})$, since $x^{\frac{3}{4}-\epsilon}$ can be taken out as a factor, and the rest tends to zero by the Riemann-Lebesgue theorem.

5. An elementary proof of van der Corput's lemma

Van der Corput's lemma states that, if $f(x)$ is real, $f'(x)$ monotonic, and $|f'(x)| < \frac{1}{2}$ in the interval (a, b), then

$$\sum_{a<n<b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,dx + O(1).$$

His proof* depends on the Fourier series for $\{x\} = x-[x]-\frac{1}{2}$, where $[x]$ is the integral part of x. The proof that follows is completely elementary. For brevity write $u_n = f(n)$. Then

$$\begin{aligned}\sum_{a<n<b} e^{2\pi i u_n} &= \sum_{a<n<b} \frac{e^{2\pi i u_n} - e^{2\pi i u_{n-1}}}{1 - e^{2\pi i(u_{n-1}-u_n)}} \\ &= \sum (e^{2\pi i u_n} - e^{2\pi i u_{n-1}})\{\tfrac{1}{2}+\tfrac{1}{2}i\cot\pi(u_{n-1}-u_n)\} \\ &= \tfrac{1}{2}i\sum_{a<n<b} (e^{2\pi i u_n} - e^{2\pi i u_{n-1}})\cot\pi(u_{n-1}-u_n) + O(1). \qquad (1)\end{aligned}$$

Now in the interval $-\frac{1}{2} < x < \frac{1}{2}$ the function

$$\phi(x) = \cot\pi x - 1/\pi x.$$

is a bounded decreasing function, its derivative being

$$-\pi\,\mathrm{cosec}^2\pi x + 1/\pi x^2 < 0.$$

Therefore, if $u_{n-1}-u_n$ decreases and $|u_{n-1}-u_n| < \frac{1}{2}$, the difference between (1) and

$$\frac{1}{2\pi i}\sum_{a<n<b} \frac{e^{2\pi i u_n} - e^{2\pi i u_{n-1}}}{u_n - u_{n-1}} \qquad (2)$$

* Van der Corput, 'Zahlentheoretische Abschatzungen': *Math. Annalen*, 84 (1921), 53–79.

is

$$\begin{aligned}&\tfrac{1}{2}i\sum(e^{2\pi iu_n}-e^{2\pi iu_{n-1}})\phi(u_{n-1}-u_n)+O(1)\\&=\tfrac{1}{2}i\sum e^{2\pi iu_n}\{\phi(u_{n-1}-u_n)-\phi(u_n-u_{n-1})\}+O(1)\\&=O\{\sum|\phi(u_{n-1}-u_n)-\phi(u_n-u_{n+1})|\}+O(1)=O(1)\end{aligned}$$

since the moduli signs may be removed and $\phi(x)$ is bounded.

It is easily verified that (2) is $\int_a^b e^{2\pi ig(x)}\,dx$, where $g(x)$ is the function linear in each interval $(n, n+1)$ contained in (a, b) and such that $g(n)=u_n$. Now

$$\left|\int_a^b(e^{2\pi if(x)}-e^{2\pi ig(x)})\,dx\right|=O\int_a^b|f(x)-g(x)|\,dx=O(A),$$

where A is the area between the convex curves $y=f(x)$ and $y=g(x)$. Again, it is easily seen from a figure that $A<\sum\Delta_n$ where Δ_n is the area of the triangle formed by the line $x=n-1$ and the lines joining the points $(n-1, u_{n-1})$, (n, u_n) and (n, u_n), $(n+1, u_{n+1})$. Therefore $\sum\Delta_n=O(1)$, since $\Delta_n=\frac{1}{2}\{u_{n+1}-2u_n+u_{n-1}\}$ and $|u_n-u_{n-1}|=O(1)$. This completes the proof.

It is possible to extend the method to functions $f(x)$ for which $\alpha<f'(x)<\beta$, but no simple argument seems to prove that

$$\sum_{\alpha<\nu<\beta}\int_a^b\{e^{2\pi if(x)}-e^{2\pi ig(x)}\}e^{-2\pi i\nu x}\,dx=O\{\log(\beta-\alpha+2)\},$$

which is required to complete the result for functions not necessarily linear in each interval $(n, n+1)$. However, it is easy to prove the desired result for functions of the type $g(x)$. The function

$$\phi(x)=\cot\pi x-\frac{1}{\pi}\sum_{\alpha<\nu<\beta}\frac{1}{x-\nu}$$

is a bounded decreasing function for $\alpha<x<\beta$ if α and β are sufficiently nearly $\frac{1}{2}$ (mod 1). Arguing as before, (1) differs from

$$\frac{1}{2\pi i}\sum_{\alpha<\nu<\beta}\sum_{a<n<b}\frac{e^{2\pi iu_n}-e^{2\pi iu_{n-1}}}{u_n-u_{n-1}-\nu} \tag{3}$$

by $$O\Big(\sum_{\alpha<\nu<\beta}\frac{1}{-\alpha+\nu}\Big)+O\Big(\sum_{\alpha<\nu<\beta}\frac{1}{\beta-\nu}\Big)=O\{\log(\beta-\alpha+2)\},$$

and (3) is equal to $\sum_{\alpha<\nu<\beta}\int_a^b e^{2\pi i\{g(x)-\nu x\}}\,dx$. This is the extended form of the lemma.

MOTIONS IN A METRIC SPACE BASED ON THE NOTION OF AREA

By E. T. DAVIES (*London*)

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When studying the equivalence of two quadratic differential forms (or, speaking geometrically, of two Riemannian spaces), Christoffel was able, by introducing an operator which has since been called covariant derivation, to write out the set of integrability conditions in a very simple and elegant form.† The same elegance is noticeable in the treatment of the integrability conditions in all equivalence problems of generalized spaces which have been considered since.‡

Later when Killing (**3**) was dealing with the problem of the *motions* (or *self-equivalence*, or *automorphisms*) admitted by a Riemannian space, he remarked§ that there was great similarity between the treatment of the integrability conditions of his equations determining a motion (which are now known as 'Killing's equations'), and the treatment given by Christoffel and Lipschitz of the integrability conditions in the equivalence problem. Killing did not, however, introduce an operator which would play, in self-equivalence problems, the role played by covariant derivation in equivalence problems.

It was Slebodziński (**4**) who succeeded in writing down the conditions of integrability for self-equivalence with the same elegance as Christoffel had done for equivalence. A systematic account of Slebodziński's operator, now known as Lie derivation, is given by Schouten (**5**).

The problem of determining the conditions for the existence of motions in spaces whose metric coefficients are functions of direction as well as position has been treated by Knebelman (**6**). Although Lie derivation is very near the surface throughout Knebelman's paper, it is never used explicitly. It is the object of this note to show how the method of Slebodziński can be extended to spaces with a generalized metric.

I shall treat the case of spaces based on the notion of area, studied for the first time by Cartan (**7**), and later in greater detail by Berwald (**8**). The treatment of integrability conditions is modelled

† (**1**), p. 57. ‡ See (**2**), Chapter VIII. § Loc. cit. 167.

on that of Knebelman for Finsler spaces. The notation will be that of Berwald.

1. Some preliminary results

If we differentiate any tensor, in the geometry which we are now considering, with respect to the components of the 'element of support', we shall have a tensor density. If we consider, for instance, the metric tensor $g_{ij}(x, u)$ which is homogeneous of degree zero in the u, and if we define

$$g_{ij;}{}^{k} = \frac{\partial g_{ij}}{\partial u_k}, \tag{1.1}$$

the law of transformation of the derivative will be

$$\bar{g}_{ij;}{}^{k}(\bar{x}, \bar{u}) = (x\bar{x}) g_{ab;}{}^{c}(x, u) \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial \bar{x}^k}{\partial x^c}, \tag{1.2}$$

where the $(x\bar{x})$ denotes the determinant of the transformation.†

For further derivations with respect to u we shall have the law of transformation

$$\bar{g}_{ij;}{}^{k_1 \ldots k_r}(\bar{x}, \bar{u}) = (x\bar{x})^r g_{ab;}{}^{c_1 \ldots c_r} \frac{\partial x^a}{\partial \bar{x}^i} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial \bar{x}^{k_1}}{\partial x^{c_1}} \cdots \frac{\partial \bar{x}^{k_r}}{\partial x^{c_r}}. \tag{1.3}$$

Similarly for the connexion parameters, with the law

$$\bar{\Gamma}_j{}^{i}{}_{k}(\bar{x}, \bar{u}) = \frac{\partial \bar{x}^i}{\partial x^c}\left[\frac{\partial^2 x^c}{\partial \bar{x}^j \partial \bar{x}^k} + \Gamma_a{}^{c}{}_{b}(x, u) \frac{\partial x^a}{\partial \bar{x}^j} \frac{\partial x^b}{\partial \bar{x}^k}\right], \tag{1.4}$$

we shall have in general

$$\bar{\Gamma}_j{}^{i}{}_{k;}{}^{h_1 \ldots h_r}(\bar{x}, \bar{u}) = (x\bar{x})^r \Gamma_a{}^{c}{}_{b;}{}^{d_1 d_2 \ldots d_r}(x, u) \frac{\partial x^a}{\partial x^j} \frac{\partial x^b}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^c} \frac{\partial \bar{x}^{h_1}}{\partial x^{d_1}} \cdots \frac{\partial \bar{x}^{h_r}}{\partial x^{d_r}}. \tag{1.5}$$

We now write

$$\overset{*}{\Gamma}_j{}^{i}{}_{k} = \Gamma_j{}^{i}{}_{k} + A_j{}^{ir}\Gamma_{r0k}$$

and

$$\overset{*}{R}_j{}^{i}{}_{kl} = \frac{\partial \overset{*}{\Gamma}_j{}^{i}{}_{k}}{\partial x^l} + \overset{*}{\Gamma}_j{}^{i}{}_{k}||^{n}\overset{*}{\Gamma}_{n0l} + \overset{*}{\Gamma}_n{}^{i}{}_{l}\overset{*}{\Gamma}_j{}^{n}{}_{k} - k|l, \tag{1.6}$$

where $k|l$ will denote the terms obtained from those already written by interchanging k and l. (These quantities are indicated by a bar in Berwald's paper.)

From the definition (1.6) we obtain immediately

$$\overset{*}{R}_j{}^{i}{}_{kl} + \text{cycl.} = 0, \tag{1.7}$$

† (2), 30.

where 'cycl.' will denote the terms obtained from the one written by a cyclic interchange of *jkl*.

We next proceed to generalize Bianchi's identity to this geometry. For this purpose we introduce the abbreviations

$$X^i||^{hk} = X^i||^h||^k, \qquad X^i|_{hk} = X^i|_h|_k,$$

$$\begin{pmatrix} i & n \\ & m \end{pmatrix} = \delta^i_m X^n + X^i||^n l_m = \delta^i_m X^n + X^i_{;}{}^n u_m,$$

and then quote the following alternation formulae

$$X^i||^{hk} - X^i||^{kh} = X^i||^h(l^k - A^k) - X^i||^k(l^h - A^h), \tag{1.8}$$

$$X^i|_{hk} - X^i|_{kh} = \begin{pmatrix} i & n \\ & m \end{pmatrix} \overset{*}{R}{}^m_{n\,hk}, \tag{1.9}$$

$$X^i|_h||^k - X^i||^k|_h = \begin{pmatrix} i & n \\ & m \end{pmatrix} \overset{*}{\Gamma}{}^m_{n\,h}||^k, \tag{1.10}$$

$$X^i|_{h;}{}^k - X^i_{;}{}^k|_h = \begin{pmatrix} i & n \\ & m \end{pmatrix} \overset{*}{\Gamma}{}^m_{n\,h;}{}^k. \tag{1.11}$$

We differentiate (1.9) covariantly with respect to x^l, write down the two equations deducible from it on cyclic interchange of h, k, l, and add so that the terms on the left-hand side can be written in groups of the type $X^i|_{hkl} - X^i|_{hlk}$. To each group we can apply (1.9) rewritten for the mixed tensor $X^i|_h$. On applying the equation (1.10) at this stage and rearranging, we finally get

$$X^m(\overset{*}{R}{}^i_{m\,hkl} - \overset{*}{\Gamma}{}^i_{m\,h}||^n \overset{*}{R}_{n0kl} + \text{cycl.}) +$$

$$+ X^i||^m(\overset{*}{R}_{m0hkl} - l_p \overset{*}{\Gamma}{}^p_{m\,h}||^n \overset{*}{R}_{n0kl} + \text{cycl.}) = 0,$$

so that, since X^i is arbitrary, we have the required identity

$$\overset{*}{R}{}^i_{m\,hkl} - \overset{*}{\Gamma}{}^i_{m\,h}||^n \overset{*}{R}_{n0kl} + \text{cycl.} = 0. \tag{1.12}$$

2. Motions

A space is said to admit *motions into itself* or *automorphisms* if it is possible to establish one or more correspondences $P \to \overline{P}$ of the space on itself in such a way that corresponding points P, $\overline{P}$ or configurations of such points are *intrinsically* (i.e. by means of the structure of the space alone) indistinguishable.† In the case of metric spaces based on the notion of area the structure is charac-

† See Robertson, (**9**), 496, and Veblen, (**10**), Ch. V, 78.

terized by the three sets of functions $g_{ij}(x, u)$, $C_{ij}{}^{k}(x, u)$ and $\Gamma_{j}^{i}{}_{k}(x, u)$. The functions $\Gamma_{j}^{i}{}_{k}$ can be replaced for the present purpose by the closely related functions $\overset{*}{\Gamma}_{j}^{i}{}_{k}$.

Expressed symbolically therefore the transformation

$$\bar{x}^{i} = x^{i} + \xi^{i}(x)\, dt \tag{2.1}$$

determines an automorphism if, *at the same time*,

$$\bar{g}_{ij}(\bar{x}, \bar{u}) = g_{ij}(\bar{x}, \bar{u}), \quad \bar{C}_{ij}\ (\bar{x}, \bar{u}) = C_{ij}{}^{k}(\bar{x}, \bar{u}), \quad \overset{*}{\bar{\Gamma}}_{i}{}^{k}{}_{j}(\bar{x}, \bar{u}) = \overset{*}{\Gamma}_{i}{}^{k}{}_{j}(\bar{x}, \bar{u}). \tag{2.2}$$

In view of the laws of transformation of the functions involved the equations (2.2) can be written

$$\left.\begin{aligned} g_{\alpha\beta}(x, u)\frac{\partial x^{\alpha}}{\partial \bar{x}^{i}}\frac{\partial x^{\beta}}{\partial \bar{x}^{j}} &= g_{ij}(\bar{x}, \bar{u}) \\ C_{\alpha\beta}{}^{\gamma}(x, u).(x\bar{x})\frac{\partial x^{\alpha}}{\partial \bar{x}^{i}}\frac{\partial x^{\beta}}{\partial \bar{x}^{j}}\frac{\partial \bar{x}^{k}}{\partial x^{\gamma}} &= C_{ij}{}^{k}(\bar{x}, \bar{u}) \\ \frac{\partial \bar{x}^{k}}{\partial x^{\gamma}}\left\{\frac{\partial^{2} x^{\gamma}}{\partial \bar{x}^{i}\partial \bar{x}^{j}} + \overset{*}{\Gamma}_{\alpha}{}^{\gamma}{}_{\beta}(x, u)\frac{\partial x^{\alpha}}{\partial \bar{x}^{i}}\frac{\partial x^{\beta}}{\partial \bar{x}^{j}}\right\} &= \overset{*}{\Gamma}_{i}{}^{k}{}_{j}(\bar{x}, \bar{u}) \end{aligned}\right\}. \tag{2.3}$$

If we now define, with reference to the transformation (2.1), the 'Lie derivative' of any geometric object $X::$ (e.g. a tensor, tensor density, connexion parameters, etc.) as

$$\Delta X:: = \lim_{dt \to 0} \frac{X::(\bar{x}, \bar{u}) - \overline{X}::(\bar{x}, \bar{u})}{dt}, \tag{2.4}$$

it is evident that equations (2.2) and (2.3) become

$$(a)\ \Delta g_{ij} = 0, \qquad (b)\ \Delta C_{ij}{}^{k} = 0, \qquad (c)\ \Delta \overset{*}{\Gamma}_{i}{}^{k}{}_{j} = 0. \tag{2.5}$$

Let us consider the expression for the Lie derivative in a few simple cases. For a contravariant vector field, for instance, we have

$$X^{i}(\bar{x}, \bar{u}) = X^{i}(x, u) + \frac{\partial X^{i}}{\partial x^{j}}\delta x^{j} + \frac{\partial X^{i}}{\partial u_{j}}\delta u_{j},$$

where $\delta x^{j} = \xi^{j}\, dt$ and where, for δu_{j}, we shall take

$$\delta u_{j} = \bar{u}_{j}(\bar{x}) - u_{j}(x) = -(u_{m}\partial_{j}\xi^{m} - u_{j}\partial_{m}\xi^{m})\, dt.$$

If we now make use of the homogeneity of degree zero of the $X^{i}(x, u)$ with respect to u, we shall have

$$\frac{\partial X^{i}}{\partial u_{j}}\delta u_{j} = -\frac{\partial X^{i}}{\partial u_{j}}u_{m}\partial_{j}\xi^{m}\, dt. \tag{2.6}$$

Similarly, $$\bar{X}^i(\bar{x},\bar{u}) = X^i(x,u)+X^j\partial_j\xi^i\,dt, \tag{2.7}$$

so that

$$\Delta X^i = \xi^j X^i|_j - X^j\xi^i|_j - X^i_{;}{}^m u_j\xi^j|_m = \xi^j X^i|_j - X^j\xi^i|_j - X^i||^m l_j\xi^j|_m, \tag{2.8}$$

$$\Delta X_i = \xi^j X_i|_j + X_j\xi^j|_i - X_{i;}{}^j u_m\xi^m|_j = \xi^j X_i|_j + X_j\xi^j|_i - X_i||^j l_m\xi^m|_j. \tag{2.9}$$

The application to the metric tensor will therefore give

$$\Delta g_{ij} = \xi^m g_{ij}|_m + g_{mj}\xi^m|_i + g_{im}\xi^m|_j - g_{ij;}{}^m u_n\xi^n|_m, \tag{2.10}$$

which, on taking account of the fact that $g_{ij}|_m = 0$, gives

$$\Delta g_{ij} = \xi_i|_j + \xi_j|_i - 2A_{ij}{}^m l_n\xi^n|_m = g_{mj}\xi^m|_i + g_{im}\xi^m|_j - g_{ij;}{}^m u_n\xi^n|_m. \tag{2.11}$$

Let us next consider the tensor density $C_{ij}{}^k = \frac{1}{2}g_{ij;}{}^k$ which is homogeneous of degree -1 in the u and of weight $+1$. For this

$$\bar{C}_{ij}{}^k(\bar{x},\bar{u}) = C_{ij}{}^k(x,u)+[\xi^m\partial_m C_{ij}{}^k - C_{ij}{}^k_{;}{}^n u_m\partial_n\xi^m - C_{ij}{}^k\partial_m\xi^m]\,dt, \tag{2.12}$$

$$\bar{C}_{ij}{}^k(\bar{x},\bar{u}) = (x\bar{x})C_{\alpha\beta}{}^\gamma(x,u)\frac{\partial x^\alpha}{\partial\bar{x}^i}\frac{\partial x^\beta}{\partial\bar{x}^j}\frac{\partial\bar{x}^k}{\partial x^\gamma}$$

$$= C_{ij}{}^k(x,u)+[C_{ij}{}^m\partial_m\xi^k - C_{mj}{}^k\partial_i\xi^m - C_{im}{}^k\partial_j\xi^m - C_{ij}{}^k\partial_m\xi^m], \tag{2.13}$$

so that

$$2\Delta C_{ij}{}^k = \Delta g_{ij;}{}^k = \xi^m\partial_m(g_{ij;}{}^k) - g_{ij;}{}^{km}u_n\partial_m\xi^n - g_{ij;}{}^m\partial_m\xi^k + \\ + g_{mj;}{}^k\partial_i\xi^m + g_{im;}{}^k\partial_j\xi^m. \tag{2.14}$$

Again, for the connexion parameters $\overset{*}{\Gamma}{}^i_{jk}$, we have

$$\Delta\overset{*}{\Gamma}{}^i_{jk} = \frac{\partial^2\xi^i}{\partial x^j\partial x^k} + \overset{*}{\Gamma}{}^i_{jm}\partial_k\xi^m + \overset{*}{\Gamma}{}^i_{mk}\partial_j\xi^m - \overset{*}{\Gamma}{}^m_{jk}\partial_m\xi^i + \\ + \xi^m\partial_m\overset{*}{\Gamma}{}^i_{jk} - \overset{*}{\Gamma}{}^i_{j;}{}^m u_n\partial_m\xi^n$$

$$= \xi^i|_{jk} + \xi^m\overset{*}{R}{}^i_{jkm} - \overset{*}{\Gamma}{}^i_{k;}{}^n u_m\xi^m|_n. \tag{2.15}$$

The following alternation formulae will be needed later:

$$\Delta(X^i|_h) - (\Delta X^i)|_h = (\delta^i_j X^k + X^i_{;}{}^k u_j)\Delta\overset{*}{\Gamma}{}^j_{kh}, \tag{2.16}$$

$$\Delta(X_i|_h) - (\Delta X_i)|_h = (-\delta^k_i X_j + X_{i;}{}^k u_j)\Delta\overset{*}{\Gamma}{}^j_{kh}, \tag{2.17}$$

and for the metric tensor

$$\Delta(g_{ij}|_h) - (\Delta g_{ij})|_h = (-\delta^k_i g_{mj} - \delta^k_j g_{im} + g_{ij;}{}^k u_m)\Delta\overset{*}{\Gamma}{}^m_{kh}, \tag{2.18}$$

so that, using $g_{ij}|_h = 0$, it becomes

$$(\Delta g_{ij})|_h = (\delta^k_i g_{mj} + \delta^k_j g_{im} - g_{ij;}{}^k u_m)\Delta\overset{*}{\Gamma}{}^m_{kh}. \tag{2.19}$$

3. Conditions of integrability

Let us now consider the equations $\Delta g_{ij} = 0$: that is

$$\xi^m \partial_m g_{ij} + g_{im}\partial_j \xi^m + g_{mj}\partial_i \xi^m - g_{ij;}{}^m u_n \partial_m \xi^n = 0. \tag{3.1}$$

On partial derivation with respect to u we get, as can be seen by comparison with (2.14),

$$\Delta g_{ij;}{}^k = 0, \tag{3.2}$$

and generally, on repeating the operation

$$\Delta g_{ij;}{}^{k_1 k_2 \ldots k_r} = 0 \quad (r = 1, 2, \ldots). \tag{3.3}$$

On covariant derivation of (3.1) we get $(\Delta g_{ij})|_h = 0$, which, by (2.19), is evidently included in the condition

$$\Delta \overset{*}{\Gamma}{}_j^i{}_k = 0. \tag{3.4}$$

Hence of the equations (2.5) determining an automorphism of the space the first conditions of integrability of the equations (a) are included in (b) and (c).

To investigate further conditions obtained on covariant derivation, we write equations (3.4) in the form

$$\xi^i|_{jk} = -\xi^m \overset{*}{R}_j{}^i{}_{km} + \overset{*}{\Gamma}{}_j^i{}_k||^n l_m \xi^m|_n \tag{3.5}$$

and consider the difference $\xi^i|_{jkl} - \xi^i|_{jlk}$. Since $\xi^i|_j$ is a mixed tensor we can apply to it the alternation formula (1.9). We can also consider the difference as obtained by covariant derivation of (3.5). Equating the results so obtained, applying the equation (1.8), then the generalization of the Bianchi identity, and finally

$$\overset{*}{R}_j{}^i{}_{kl}||^n = \overset{*}{\Gamma}{}_j^i{}_k||^n|_l + \overset{*}{\Gamma}{}_j^i{}_k||^q l_p \overset{*}{\Gamma}{}_q^p{}_l||^n - k|l, \tag{3.6}$$

we shall have, on rearranging,

$$\xi^m \overset{*}{R}_j{}^i{}_{kl}|_m - \overset{*}{R}_j{}^i{}_{kl}||^n l_m \xi^m|_n - \overset{*}{R}_j{}^m{}_{kl}\xi^i|_m + \overset{*}{R}{}^i_{m\,kl}\xi^m|_j + \\ + \overset{*}{R}_j{}^i{}_{ml}\xi^m|_k + \overset{*}{R}_j{}^i{}_{km}\xi^m|_l = 0, \tag{3.7}$$

which is simply

$$\Delta \overset{*}{R}_j{}^i{}_{kl} = 0. \tag{3.8}$$

Further covariant derivation and application of (3.4) will lead, as can easily be seen from (2.16), to

$$\Delta(\overset{*}{R}_j{}^i{}_{kl}|_{m_1 m_2 \ldots m_s}) = 0 \quad (s = 1, 2, \ldots). \tag{3.9}$$

We must further consider the equations obtained (i) by partial

derivation with respect to u_i of (3.4), (3.8), (3.9), (ii) by covariant derivation of (3.2) and (3.3).

The partial derivation with respect to (3.4) will give the set

$$\Delta \overset{*}{\Gamma}{}^{i}_{j\,k;}{}^{l_1 l_2 \ldots l_q} = 0 \quad (q = 1, 2, \ldots). \tag{3.10}$$

Since the left-hand member of this equation is homogeneous of degree $-q$ in the u, we shall have the Eulerian relation

$$u_m \Delta \overset{*}{\Gamma}{}^{i}_{j\,k;}{}^{l_1 \ldots l_{q-1} m} = -q \Delta \overset{*}{\Gamma}{}^{i}_{j\,k;}{}^{l_1 l_2 \ldots l_{q-1}}, \tag{3.11}$$

so that the final equations of the set (3.10) include all the preceding ones (i.e. for smaller values of q). Thus we may rewrite (3.10) as

$$\Delta \overset{*}{\Gamma}{}^{i}_{j\,k;}{}^{l_1 l_2 \ldots l_q} = 0. \tag{3.12}$$

For the covariant derivation of (3.2), which is equivalent to $\Delta C_{ij}{}^{k} = 0$, we first of all quote the rule for the interchange of the order of the two operators $;^j$ and $|_j$, as applied to the metric tensor g_{ij}. On referring to (1.11) it is

$$g_{ij;}{}^{k}{}_{|l} - g_{ij}{}_{|l;}{}^{k} = \overset{*}{\Gamma}{}_{m}{}^{n}{}_{l;}{}^{k} \{\delta_i^m g_{nj} + \delta_j^m g_{in} - g_{ij;}{}^{m} u_n\}. \tag{3.13}$$

Then, since the operator Δ is, for a motion, permutable both with partial derivation with respect to u and with covariant derivation, we shall have $(\Delta g_{ij;}{}^{k})_{|l}$ expressed in terms of the left-hand sides of equations (3.1), (3.2), (3.4), and (3.12). Hence, since Δu_l and $\Delta \delta_k^i$ also vanish, no new conditions are obtained in this way. Similarly it may be proved that the covariant derivation of the set (3.3) at any stage leads to no new conditions.

Since the results of the covariant derivation of (3.1) are included in (3.4), we can confine ourselves to the latter in the sequel. We have already seen that continued partial derivation of (3.4) leads to the set (3.12). Covariant derivation of (3.12) leads to a set

$$\Delta \overset{*}{\Gamma}{}^{i}_{j\,k;}{}^{l_1 l_2 \ldots l_q}{}_{|m_1 m_2 \ldots m_t} = 0 \quad (t = 0, 1, 2, \ldots), \tag{3.14}$$

which cannot be expressed in terms of any others already given. A partial derivation applied at any stage to the set (3.14) would provide no new results, as can easily be verified by considering the application of the permutation formula (1.11). The result of a partial derivation would always be expressible in terms of a further covariant derivation of the type (3.14) itself.

In our search for new conditions of integrability it only remains to consider partial derivation applied to (3.8) or at any stage of (3.9). We first remark that

$$\overset{*}{R}_{m\ hk;}^{\ i\ \ \ \ \ l} = \overset{*}{\Gamma}_{m\ h;}^{\ i\ \ \ l}|_k + \overset{*}{\Gamma}_{m\ h;}^{\ i\ \ \ n} u_p \overset{*}{\Gamma}_{n\ k;}^{\ p\ \ \ l} + \overset{*}{\Gamma}_{m\ h;}^{\ i\ \ \ l} \overset{*}{\Gamma}_{n\ k}^{\ n} - h|k, \qquad (3.15)$$

so that, on applying the operator Δ to it and taking account of the permutability of the operators involved, we shall have $(\Delta \overset{*}{R}_{m\ hk)}^{\ i\ \ \ \ \ ;l}$ expressed in terms of the left-hand sides of (3.4), (3.10), (3.14), so that no new conditions will be obtained by partial derivation with respect to u_i of (3.8).

Finally we consider the partial derivation of (3.9) for $s = 1$. We first of all apply (1.11) to the tensor $\overset{*}{R}_{j\ kl}^{\ i}$, and then apply Δ to the resulting equation. This will serve to express $(\Delta \overset{*}{R}_{j\ kl|m)}^{\ i\ \ \ \ \ ;n}$ in terms of

(i) the left-hand sides of (3.4) and (3.12),

(ii) $\Delta \overset{*}{R}_{j\ kl;}^{\ i\ \ \ \ m}$, which we have just dealt with,

(iii) $(\Delta \overset{*}{R}_{j\ kl;}^{\ i\ \ \ \ n})|_m$, which, on applying Δ to (3.15) and then deriving covariantly, can be seen to be expressible in terms of the left-hand sides of (3.4), (3.12), (3.14).

This argument can be repeated at every stage of the equations (3.9), i.e. for $s = 2, 3, \ldots$. Hence no new conditions are obtainable by partial derivation with respect to u_i of the equations (3.8) or (3.9).

In conclusion therefore we can state that the conditions of integrability of (2.5 a, b, c) are all included in the sets (3.3), in which the last set include all the preceding in view of the remarks made about the set (3.10), (3.9), and (3.14). If these sets of equations in the n^2+n unknowns ξ^i and $\partial\xi^i/\partial x^j$ form a compatible system, then they determine, together with (2.5), an automorphism of the space.

We conclude by pointing out that, if $\xi^i_{(\alpha)}$ ($\alpha = 1, 2, \ldots, r$) are the components of r linearly independent motions, the components of the alternant of any two of them satisfy the same equations. This is done immediately by the method used by Knebelman,† but our purpose here is to point out how the use of Lie derivation simplifies the calculations.

† Loc. cit. § 8.

Let $\xi^i_{(\alpha)}$ and $\xi^i_{(\beta)}$ define two linearly independent motions in the space considered; then the components of their alternant are

$$\xi^i = \xi^m_{(\beta)}\xi^i_{(\alpha)}|_m - \xi^m_{(\alpha)}\xi^i_{(\beta)}|_m$$

i.e., in our notation, $\xi^i = \Delta_\beta \xi^i_{(\alpha)}$.

We wish to prove that, if the 'structure functions' of the space admit Δ_α and Δ_β, they will also admit Δ (referring to $\xi^i = \Delta_\beta \xi^i_{(\alpha)}$), i.e. they will admit an r-parameter continuous group of motions. The point will be sufficiently illustrated by considering the case of the functions $\overset{*}{\Gamma}{}^i_{jk}$. If $\Delta_\alpha \overset{*}{\Gamma}{}^i_{jk} = 0$ and $\Delta_\beta \overset{*}{\Gamma}{}^i_{jk} = 0$, then $\Delta \overset{*}{\Gamma}{}^i_{jk} = 0$. To prove this we write $\Delta_\alpha \overset{*}{\Gamma}{}^i_{jk} = 0$ in the form

$$\xi^i_{(\alpha)}|_{jk} + \xi^m_{(\alpha)} \overset{*}{R}{}^i_{jkm} - \overset{*}{\Gamma}{}^i_{jk;}{}^n u_m \xi^m_{(\alpha)}|_n = 0$$

and apply Δ_β to this equation. Since $\xi^i_{(\beta)}$ determines a motion, the operation Δ_β is permutable with both partial and covariant derivation, so that we may write

$$\begin{aligned}(\Delta_\beta \xi^i_{(\alpha)})|_{jk} + (\Delta_\beta \xi^m_{(\alpha)}) \overset{*}{R}{}^i_{jkm} - \overset{*}{\Gamma}{}^i_{jk;}{}^n u_m (\Delta_\beta \xi^m_{(\alpha)})|_n \\ = -\xi^m_{(\alpha)} \Delta_\beta \overset{*}{R}{}^i_{jkm} + (\Delta_\beta \overset{*}{\Gamma}{}^i_{jk;}{}^n) u_m \xi^m_{(\alpha)}|_n.\end{aligned}$$

The right-hand side of this vanishes on account of equations (3.8) and (3.12), which are by hypothesis satisfied by $\xi^i_{(\beta)}$. Hence the left-hand side also vanishes. But the left-hand side is $\Delta \overset{*}{\Gamma}{}^i_{jk}$, the vanishing of which is therefore established.

REFERENCES

1. E. B. Christoffel, *J. für Math.* 70 (1869), 46–70.
2. T. Y. Thomas, *The Differential Invariants of Generalized Spaces* (Cambridge 1934).
3. W. Killing, *J. für Math.* 109 (1892), 121–86.
4. W. Slebodziński, *Prace Mat. Fiz.*, Warsaw, 39 (1932), 55–62.
5. J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie* I (Gröningen 1935).
6. M. S. Knebelman, *American J. of Math.* 51 (1929), 527–64.
7. E. Cartan, 'Les espaces métriques fondés sur la notion d'aire', *Actualités Scientifiques et Industrielles*, 72 (Paris 1933).
8. L. Berwald, *Acta Math.* 71 (1939), 191–248.
9. H. P. Robertson, *Annals of Math.* (2) 33, 496–521.
10. O. Veblen, *Invariants of Quadratic Differential Forms* (Cambridge 1933).

THE SPECTRAL ANALYSIS OF SELF-ADJOINT OPERATORS

By J. L. B. COOPER (*Oxford*)

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1. THE object of this paper is to prove the basic theorem on the analysis of self-adjoint operators in Hilbert space, that concerning the expression of a self-adjoint operator in terms of a set of projection operators.

The theorem was proved in its present form by von Neumann,† and independent proofs were given by Stone‡ and others. The notation used in this paper is that of Stone's book.

As proved by von Neumann and Stone, the theorem reads as follows:§

For a given self-adjoint operator H there exists a set of projection operators $E(\lambda)$ $(-\infty < \lambda < \infty)$ with the properties:

(i) *when* $\lambda \to -\infty$, $E(\lambda)f \to 0$; *when* $\lambda \to \infty$, $E(\lambda)f \to f$; *when* $\lambda \to \lambda_0$ $(\lambda > \lambda_0)$, $E(\lambda)f \to E(\lambda_0)f$, *for every* f;

(ii) *if* $\lambda' \leqslant \lambda''$, $E(\lambda')E(\lambda'') = E(\lambda')$;

(iii) *the integral*

$$\int_{-\infty}^{\infty} \lambda^2 \, d(\|E(\lambda)f\|^2)$$

defines the domain of Hf, *which has meaning if and only if the integral is convergent*: *in that case, for all* g,

$$(Hf, g) = \int_{-\infty}^{\infty} \lambda \, d(E(\lambda)f, g).$$

(*The last integral is absolutely convergent if the first is finite.*)

In this paper a radically different method is used, and, in the first place, a different theorem is proved from that of von Neumann and Stone. In Theorem I of this paper the hypothesis of their theorem, that H is self-adjoint, is replaced by the hypothesis that H is symmetric and that the Schrödinger equation involving H—

† See *Math. Annalen*, 102 (1929), 49–131. Also *Mathematische Grundlagen der Quantenmechanik*, hereinafter referred to as 'M.G.'.

‡ Stone, 'Linear Transformations in Hilbert Space': *American Math. Soc. Colloquium Publications*. (Hereinafter referred to as 'Stone'.)

§ See M.G. II. 7 and Stone, Chapter V.

equation (1) of this paper—has a solution for all t. The existence of projection operators satisfying conditions (i), (ii), (iii) is proved with this assumption. It is clear that this new assumption has greater physical significance than has the assumption that H is self-adjoint; the latter property does not convey any direct physical notion at all.

The method used in the proof of Theorem I is the deduction of the spectral resolution of the operator from the spectral analysis of the solution of the Schrödinger equation considered as a function of time. Mathematically the method is similar to that used by Titchmarsh in his papers on eigenfunctions,† but differs from his in that the real-variable theory of Fourier-Stieltjes integrals is used, whereas Titchmarsh uses the Fourier transform in the complex plane, and relies rather on the properties of the resolvent operator than on those of the solution of the Schrödinger equation. The present method seems more applicable to the proof of the general theorem, but less suitable for the deduction of particular expansions.

The results of this paper also have a connexion with the work of Stone and others on groups of unitary transformations. Stone uses the theory of the resolvent operator and the existence of a spectral resolution of a self-adjoint operator to prove the existence of spectral resolution for a group of unitary operators. The present article, on the other hand, proves the existence of the spectral resolution of the solution of the Schrödinger equation (which is a unitary transformation of its initial value) directly, and deduces the spectral resolution of the operator H from that. Also it differs in that real-variable theory only is used.‡

In order to link Theorem I with the theorem of von Neumann and Stone cited above, a further section of the paper gives a direct proof that, if H is a self-adjoint operator, then the equation (1) has a solution. Together these two theorems constitute a proof of the theorem of von Neumann and Stone.§

† *J. of London Math. Soc.* (2) 14 (1939), 274–8, and *Quart. J. of Math.* (Oxford), 1940 *passim.*

‡ B. O. Koopman, *Proc. Nat. Acad. of Science, U.S.A.* 17 (1931), 315–18; M. H. Stone, ibid. 16 (1930), 172–5; M. H. Stone, *Annals of Math.* 33 (1932), 643–8; J. von Neumann, ibid. 567–73.

§ Since the above was written, it has been found possible to prove Theorem II without the employment of complex-variable methods, or of the resolvent operator. These results, which give a new basis for the theory of symmetric operators, will be published in a paper on 'The Schrödinger equation for symmetric operators'.

2. We consider a linear symmetric operator defined in a Hilbert space R and subject to the following condition:

(a) If D is the domain of the linear symmetric operator H, then there exists a linear manifold D' which is an everywhere dense subset of D, such that for every element ϕ of D' a function $\psi(t)$ $(-\infty < t < \infty)$ can be found which, for every t, is a point of D, whose derivative $d\psi/dt$ exists for all t and is an element of R, and which satisfies the equations

$$\psi(0) = \phi, \qquad \frac{1}{i}\frac{d\psi}{dt} = H\psi(t). \tag{1}$$

The theorem to be proved is

THEOREM I. *If H is a closed† linear symmetric operator satisfying condition (a), then there is a set of projection operators satisfying the conditions* (i), (ii), (iii).

In what follows we shall in the first place restrict attention to points ϕ in D', and later extend to other points in R.

Let ϕ be any point in D', and $\psi(t)$ be defined by (1). Then, for any values of t and τ,

$$\frac{d}{dt}\{\psi(t+\tau), \psi(t)\} = \left(\frac{d\psi(t+\tau)}{dt}, \psi(t)\right) + \left(\psi(t+\tau), \frac{d\psi(t)}{dt}\right)$$

$$= i\{H\psi(t+\tau), \psi(t)\} - i\{\psi(t+\tau), H\psi(t)\} = 0,$$

so that $\{\psi(t+\tau), \psi(t)\}$ is a function of τ only, say $\chi(\tau)$. In particular

$$\{\psi(t), \psi(t)\} = \|\psi(t)\|^2 = \|\psi(0)\|^2 = \chi(0). \tag{2}$$

Choose a set of functions ϕ_n lying in D' and forming a complete normal orthogonal set in R, and let

$$\{\psi(t), \phi_n\} = h_n(t),$$

so that

$$\psi(t) = \sum h_n(t)\phi_n.$$

From (2)

$$\sum |h_n(t)|^2 = \chi(0) = \text{constant}, \tag{3}$$

and generally

$$\sum h_n(t+\tau)\overline{h_n(t)} = \{\psi(t+\tau), \psi(t)\} = \chi(\tau). \tag{4}$$

† The assumption that H is closed is used only in the proof of the assertion that Hf exists if the integral $\int_{-\infty}^{\infty} \lambda^2 d(\|E(\lambda)f\|^2)$ is convergent.

The function $h_n(t)$ is a differentiable function of t for

$$\frac{1}{i}\frac{d}{dt}h_n(t) = \frac{1}{i}\frac{d}{dt}\{\psi(t), \phi_n\} = \{H\psi(t), \phi_n\}$$

since $\psi(t)$ is in D' and satisfies (1). From (3), and because of the differentiability of $h_n(t)$, it follows that $\{h_n(t)-h_n(0)\}/t$ is $L^2(-\infty, \infty)$, and hence the integral

$$V_n(u) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\{h_n(t)-h_n(0)\}\frac{e^{-iut}}{-it}\,dt \tag{5}$$

exists in mean square over $(-\infty, \infty)$.† Also

$$\int_{-\infty}^{\infty}|V_n(u)|^2\,du = \int_{-\infty}^{\infty}\frac{|h_n(t)-h_n(0)|^2}{t^2}\,dt,$$

and

$$\int_{-\infty}^{\infty}\{\sum|V_n(u)|^2\}\,du = \int_{-\infty}^{\infty}\{\sum|h_n(t)-h_n(0)|^2\}\frac{dt}{t^2},$$

provided that the right-hand side is finite. That the integral over any interval not including $t = 0$ is finite follows from (3). At $t = 0$,

$$h_n(t)-h_n(0) = \{\psi_n(t)-\psi(0), \phi_n\}.$$

and so

$$\frac{\sum|h_n(t)-h_n(0)|^2}{t^2} = \frac{\|\psi(t)-\psi(0)\|^2}{t^2}.$$

Now

$$\frac{\psi(t)-\psi(0)}{t} \to \frac{d\psi}{dt} = iH\phi,$$

and therefore, as $t \to 0$,

$$\sum\frac{|h_n(t)-h_n(0)|^2}{t^2} \to -\|H\phi\|^2,$$

which exists since ϕ is in D. Hence the convergence of the integral is proved. If, then, we put

$$\Phi(u) = \sum V_n(u)\phi_n, \tag{6}$$

$\Phi(u)$ is defined as a function in R for almost all u. The equation (6) may be written

$$\Phi(u) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\{\psi(t)-\psi(0)\}\frac{e^{-iut}}{-it}\,dt. \tag{7}$$

† See Titchmarsh, *Fourier Integrals* (1937), Theorem 48.

Now, for almost all u and v,

$$\{\Phi(u),\Phi(v)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\psi(t)-\psi(0),\psi(\tau)-\psi(0)\}\frac{e^{-iut+iv\tau}}{(-it)(i\tau)}\,dtd\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\chi(t-\tau)-\chi(t)-\chi(-\tau)+\chi(0)\}\frac{e^{-iut+iv\tau}}{(-it)(i\tau)}\,dtd\tau, \tag{8}$$

where the double infinite integral means

$$\lim_{\substack{T\to\infty\\ \mu\to\infty}}\int_{-T}^{T}\int_{-\mu}^{\mu} dtd\tau.$$

The function $\chi(\tau) = \{\psi(t+\tau),\psi(t)\}$ is a positive definite function,† for it is continuous and bounded, satisfies

$$\overline{\chi(-t)} = \chi(t),$$

and, for any set of numbers ρ_m, t_m,

$$\sum_{\mu}\sum_{\nu}\rho_\mu\bar{\rho}_\nu\chi(t_\mu-t_\nu) = \sum\sum\rho_\mu\bar{\rho}_\nu\{\psi(t_\mu),\psi(t_\nu)\}$$
$$= \|\sum\rho_\mu\psi(t_\mu)\|^2 \geqslant 0.$$

Hence $\chi(t)$ can be expressed as a Fourier-Stieltjes integral‡

$$\chi(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\,dP(\lambda), \tag{9}$$

where $P(\lambda)$ increases monotonically over $(-\infty,\infty)$ and

$$P(\infty)-P(-\infty) = \chi(0).$$

Substituting (9) in (8), we get

$$\{\Phi(u),\Phi(v)\}$$
$$= \frac{1}{2\pi}\lim_{\substack{T\to\infty\\ \mu\to\infty}}\int_{-T}^{T}\int_{-\mu}^{\mu}\{\chi(t-\tau)-\chi(t)-\chi(-\tau)+\chi(0)\}\frac{e^{-iut+iv\tau}}{(-it)(i\tau)}\,dtd\tau$$

$$= \frac{1}{2\pi}\lim\int_{-T}^{T}\int_{-\mu}^{\mu}\frac{e^{-iut+iv\tau}}{(-it)(i\tau)}\,dtd\tau\int_{-\infty}^{\infty}\{e^{i\lambda(t-\tau)}-e^{i\lambda t}-e^{-i\lambda\tau}+1\}\,dP(\lambda)$$

$$= \frac{1}{2\pi}\lim\int_{-\infty}^{\infty}dP(\lambda)\int_{-T}^{T}\int_{-\mu}^{\mu}\frac{e^{-iut+iv\tau}}{(-it)(i\tau)}\{e^{i\lambda(t-\tau)}-e^{i\lambda t}-e^{-i\lambda\tau}+1\}\,dtd\tau.$$

† See Bochner, *Vorlesungen über Fourier'sche Integralen*, 74.

‡ Bochner, loc. cit. 76, Satz 23.

The integrals in t and τ are boundedly convergent, so that we can proceed to the limit under the integral sign. This gives

$$\{\Phi(u),\Phi(v)\} = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} \{\operatorname{sgn}(u-\lambda)\operatorname{sgn}(v-\lambda) - \operatorname{sgn} u \operatorname{sgn}(v-\lambda) - \operatorname{sgn} v \operatorname{sgn}(u-\lambda) + \operatorname{sgn} u \operatorname{sgn} v\}\, dP(\lambda). \quad (10)$$

From equation (5) we get, by the inversion theorem for Fourier transforms,†

$$\frac{h_n(t)-h_n(0)}{-it} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{iut} V_n(u)\, du, \quad (11)$$

and so, on integrating by parts, we have formally

$$h_n(t)-h_n(0) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{iut}\, dV_n(u),$$

(a step which is justified if $V_n(u)$ is of bounded variation). Hence

$$h_n(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{iut}\, dW_n(u),$$

where
$$W_n(u) = \begin{cases} V_n(u) & (u < 0), \\ V_n(u)+\sqrt{(2\pi)}h_n(0) & (u \geqslant 0). \end{cases}$$

We are thus led to consider the function

$$\Psi(u) = \sum \phi_n W_n(u) = \left\{ \begin{matrix} \Phi(u) & (u < 0) \\ \Phi(u)+\sqrt{(2\pi)}\psi(0) & (u \geqslant 0) \end{matrix} \right\}, \quad (7')$$

i.e.
$$\Psi(u) = \Phi(u)+\sqrt{(\tfrac{1}{2}\pi)}(1+\operatorname{sgn} u)\psi(0),$$

for which we have shown formally that

$$\psi(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{iut}\, d\Psi(u). \quad (12)$$

Now

$$\{\Psi(u),\Psi(v)\} = \{\Phi(u),\Phi(v)\} + \sqrt{(\tfrac{1}{2}\pi)}[(1+\operatorname{sgn} v)\{\Phi(u),\psi(0)\}+(1+\operatorname{sgn} u)\{\psi(0),\Phi(v)\}] + \tfrac{1}{2}\pi(1+\operatorname{sgn} u)(1+\operatorname{sgn} v)\chi(0).$$

† See Titchmarsh, *Theory of Fourier Integrals*, Theorem 48.

We have

$$\{\Phi(u), \psi(0)\} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} [\chi(t)-\chi(0)] \frac{e^{-iut}}{-it}\, dt$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{e^{-iut}}{-it}\, dt \int_{-\infty}^{\infty} (e^{i\lambda t}-1)\, dP(\lambda)$$

$$= \sqrt{(\tfrac{1}{2}\pi)} \int_{-\infty}^{\infty} \{\operatorname{sgn}(u-\lambda)-\operatorname{sgn} u\}\, dP(\lambda).$$

Similarly,

$$\{\psi(0), \Phi(v)\} = \sqrt{(\tfrac{1}{2}\pi)} \int_{-\infty}^{\infty} \{\operatorname{sgn}(v-\lambda)-\operatorname{sgn} v\}\, dP(\lambda).$$

Here the changes of order of integration are again justified by the bounded convergence of the integral in t. We now have

$$\begin{aligned}
&\{\Psi(u), \Psi(v)\} \\
&= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} dP(\lambda)\, [\{\operatorname{sgn}(u-\lambda)-\operatorname{sgn} u\}\{\operatorname{sgn}(v-\lambda)-\operatorname{sgn} v\}+ \\
&\qquad +(1+\operatorname{sgn} v)\{\operatorname{sgn}(u-\lambda)-\operatorname{sgn} u\}+ \\
&\qquad +(1+\operatorname{sgn} u)\{\operatorname{sgn}(v-\lambda)-\operatorname{sgn} v\}+ \\
&\qquad +(1+\operatorname{sgn} u)(1+\operatorname{sgn} v)] \\
&= \tfrac{1}{2}\pi \int_{-\infty}^{\infty} dP(\lambda)\, \{\operatorname{sgn}(u-\lambda)+1\}\{\operatorname{sgn}(v-\lambda)+1\} \\
&= 2\pi \int_{-\infty}^{\min(u,v)} dP(\lambda).
\end{aligned}$$

This is an important result: it shows that, if $u > v$,

$$\{\Psi(u), \Psi(v)\} = \{\Psi(v), \Psi(v)\}$$

$$= \|\Psi(v)\|^2, \tag{13}$$

a function of v only, and that

$$\|\Psi(v)\|^2 = 2\pi \int_{-\infty}^{v} dP(\lambda). \tag{14}$$

The result (13) can be extended. Let $\Psi_1(u)$ be a function Ψ derived from an initial function ϕ_1, and let $\Psi_2(u)$ be derived from ϕ_2. Then

$\Psi_1 \pm i\Psi_2$ are similarly derived from $\phi_1 \pm i\phi_2$ and $\Psi_1 \pm \Psi_2$ from $\phi_1 \pm \phi_2$. Now

$$\begin{aligned} 4\{\Psi_2(u), \Psi_1(v)\} = &\{\Psi_1(u)+\Psi_2(u), \Psi_1(v)+\Psi_2(v)\}- \\ &-\{\Psi_1(u)-\Psi_2(u), \Psi_1(v)-\Psi_2(v)\}- \\ &-i[\{\Psi_1(u)+i\Psi_2(u), \Psi_1(v)+i\Psi_2(v)\}- \\ &-\{\Psi_1(u)-i\Psi_2(u), \Psi_1(v)-i\Psi_2(v)\}]. \end{aligned}$$

Since the expression on the right is a function of v only, so is that on the left, and therefore

$$\{\Psi_2(u), \Psi_1(v)\} = \{\Psi_2(v), \Psi_1(v)\}.$$

It follows immediately that

$$\{\Psi_2(u)-\Psi_2(v), \Psi_1(v)\} = 0: \tag{15}$$

in other words, any function of the form $\Psi_2(u)-\Psi_2(v)$ is orthogonal to any function of the form $\Psi_1(v)$ $(u > v)$.

A rigorous proof of the formula (12) can now be given. It is necessary first of all to show that the Stieltjes integral has a meaning over a finite interval (λ, μ). To do this, consider sums of the form

$$\sum_{n=1}^{\infty} e^{iu_n t}[\Psi(u_n)-\Psi(u_{n-1})] = S_N \quad \text{say},$$

where

$$\lambda = u_0 < u_1 < \ldots < u_N = \mu, \qquad \max(u_n - u_{n-1}) < \delta.$$

We must examine how far S_N can be changed by introducing further subdivisions of the interval (u_{n-1}, u_n). It is clear that the contributions of individual intervals to the sums of form S_N are mutually orthogonal, and, if the interval (u_m, u_m+1) is subdivided in any way, the contribution of the interval to the sum will be changed by the factor multiplying the term $[\Psi(u'_m)-\Psi(u'_{m-1})]$ for any sub-interval being changed from $e^{iu_m t}$ to $e^{iu'_m t}$. The change in the contribution of any factor has a magnitude less than

$$|e^{i\delta t}-1|\,\|\Psi(u'_m)-\Psi(u'_{m-1})\|$$

and, since the terms are mutually orthogonal, the square of the magnitude of the change in S_N is the sum of the squares of the changes of these individual contributions. It follows that S_N changes by less than

$$|e^{i\delta t}-1|\,\|\Psi(\mu)-\Psi(\lambda)\|$$

and this tends to zero as δ tends to zero. The existence of the Stieltjes integral is thus proved.

The formal demonstration of (12) can now be made rigorous by justifying the integration by parts. From (11), summing over all the ϕ_n, we get

$$\frac{\psi(t)-\psi(0)}{-it} = \lim_{\lambda\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-\lambda}^{\lambda}\Phi(u)e^{iut}\,du.$$

Now

$$\sum_{n=1}^{N} e^{iu_nt}[\Phi(u_n)-\Phi(u_{n-1})]$$
$$= \sum_{n=0}^{N}\Phi(u_n)(e^{iu_nt}-e^{iu_{n+1}t})-\Phi(u_0)e^{iu_0t}+\Phi(u_n)e^{iu_nt}.$$

The left-hand sum tends to the Stieltjes integral

$$\int_{\lambda}^{\mu} e^{iut}\,d\Phi(u),$$

and the right-hand expression to

$$-it\int_{\lambda}^{\mu}\Phi(u)e^{iut}\,du\ -\Phi(\lambda)e^{i\lambda t}+\Phi(\mu)e^{i\mu t}.$$

Hence

$$\int_{-\lambda}^{\lambda} e^{iut}\,d\Phi(u) = -it\int_{-\lambda}^{\lambda}\Phi(u)e^{iut}\,du\ +\Phi(\lambda)e^{i\lambda t}-\Phi(-\lambda)e^{-i\lambda t},$$

and, since $\Phi(\lambda)\to 0$ as $\lambda\to\pm\infty$ because of equation (10), the formula (12) follows immediately.

Now $\Phi(u)\to 0$ as $u\to\pm\infty$, and so, by (7'), $\Psi(u)\to 0$ as $u\to-\infty$ and $\Psi(u)\to\sqrt{(2\pi)}\phi$ as $u\to\infty$. Hence, as a particular case of (12) when $t=0$,

$$\phi = \psi(0) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} d\Psi(u) = \lim_{u\to\infty}\Psi(u)/\sqrt{(2\pi)}. \tag{16}$$

It has now been shown that the equations (1), (7), (7') result in defining for every element ϕ in D' a function $\Psi(u)$ for any u in $(-\infty,\infty)$ satisfying (16) and such that $\Psi_1(u)-\Psi_2(u)$ is orthogonal to $\Psi_2(v)$ if $u>v$, and Ψ_1, Ψ_2 correspond to any elements ϕ_1 and ϕ_2 in D'. It is clear from these properties that

$$\|\Psi(u)\| \leqslant \sqrt{(2\pi)}\phi;$$

so the operator $T(u)$ which transforms an element ϕ of D' into $\Psi(u)$ is bounded. Since the transformation is defined for the domain D', which is everywhere dense in R, its closed linear extension exists everywhere in R, and defines for every ϕ of R a function $\Psi(u)$ satis-

fying equations (13)–(16). There will therefore be no further need to confine attention to elements of D'.

For any ϕ of R, let

$$\Psi(u) = T(u)\phi, \qquad \Psi_1(u) = T(u)\Psi(v).$$

Then, by (16), $$\sqrt{(2\pi)}\Psi(v) = \lim_{u\to\infty} \Psi_1(u),$$

and so $$\sqrt{(2\pi)}\Psi(v) - \Psi_1(v) = \lim_{u\to\infty}[\Psi_1(u) - \Psi_1(v)].$$

The left-hand side is orthogonal to $\Psi_1(u) - \Psi_1(v)$ for all $u > v$, and hence both sides must be zero: thus

$$\Psi_1(v) = \sqrt{(2\pi)}\Psi(v),$$

and, for all $u \geqslant v$, $$\Psi_1(u) = \Psi_1(v),$$

similarly, for any ϕ in R, and $u \geqslant v$,

$$T(u)[T(v)\phi] = \sqrt{(2\pi)}T(v)\phi.$$

It follows that $\{T(u)/\sqrt{(2\pi)}\}^2 = \{T(u)/\sqrt{(2\pi)}\}$ and hence that $T(u)/\sqrt{(2\pi)}$ is a projection operator:

$$T(u) = \sqrt{(2\pi)}E(u), \qquad \Psi(u) = \sqrt{(2\pi)}E(u)\phi,$$

and that, for $u > v$, $$E(u)E(v) = E(v).$$

We can now show that the projection operators $E(\lambda)$ satisfy the requirements (i), (ii), (iii) of § 1, and so solve the eigenvalue problem of the operator H.

Of these (i) is satisfied provided that the operators $E(\lambda)$ are defined at points of discontinuity to be equal to the right-hand limit, for it has already been shown that $\Psi(\lambda) \to 0$ as $\lambda \to -\infty$, and that $\Psi(\lambda) \to \sqrt{(2\pi)}\phi$ as $\lambda \to \infty$; (ii) is satisfied because of (13) and (15).

To prove (iii) satisfied for any ϕ, let

$$\phi_\lambda = [\Psi(\lambda) - \Psi(-\lambda)]/\sqrt{(2\pi)}.$$

Then, if we define the $\psi_\lambda(t)$, $\Psi_\lambda(u)$ corresponding to ϕ_λ, we see that

$$\Psi_\lambda(u) = \begin{cases} 0 & (u < -\lambda), \\ \Psi(u) - \Psi(-\lambda) & (-\lambda \leqslant u < \lambda), \\ \Psi(\lambda) - \Psi(-\lambda) & (u \geqslant \lambda), \end{cases}$$

and

$$\psi_\lambda(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{iut}\, d\Psi_\lambda(u)$$

$$= \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} e^{iut}\, d\Psi(u).$$

Now $$H\psi_\lambda(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} ue^{itu}\, d\Psi(u),$$

since $$H\psi_\lambda(t) = \frac{1}{i}\frac{d\psi_\lambda(t)}{dt}$$

and since we can differentiate under the integral sign. In particular,

$$H\phi_\lambda = \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} u\, d\Psi(u). \tag{17}$$

$H\phi$ will exist if and only if $H\phi_\lambda$ tends to a limit as $\lambda \to \infty$, and, if the limit exists, $H\phi$ will be equal to the limit. Now

$$H\phi_\lambda = \frac{1}{\sqrt{(2\pi)}} \lim_{n\to\infty} \sum_{1}^{n} u_n[\Psi(u_m) - \Psi(u_{m-1})],$$

where $-\lambda = u_0 < u_1 < u_2 \ldots < u_n = \lambda$ are division points whose maximum distance apart tends to zero as n tends to infinity. Since the $[\Psi(u_m) - \Psi(u_{m-1})]$ are mutually orthogonal,

$$\| \sum u_m[\Psi(u_m) - \Psi(u_{m-1})]\|^2 = \sum |u_m|^2 \|\Psi(u_m) - \Psi(u_{m-1})\|^2$$
$$\to \int_{-\lambda}^{\lambda} u^2\, d\|\Psi(u)\|^2.$$

Hence $H\phi$ exists if and only if

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^2\, d\|\Psi(\lambda)\|^2 = \int_{-\infty}^{\infty} \lambda^2\, dP(\lambda)$$

is finite, and in that case the integral is equal to $\|H\phi\|^2$. Also, from (17),

$$(H\phi_\lambda, g) = \frac{1}{\sqrt{(2\pi)}} \int_{-\lambda}^{\lambda} u\, d\{\Psi(u), g\},$$

for any g. Making $\lambda \to \infty$, we see that, if $H\phi$ exists,

$$(H\phi, g) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} u\, d\{\Psi(u), g\}$$

$$= \int_{-\infty}^{\infty} u\, d\{E(u)\phi, g\}.$$

Thus $E(u)$ satisfies the requirements (i), (ii), (iii) and the existence of a spectral resolution for H is proved.

3. The theorem to be proved in this section, when added to Theorem I of the last section, gives a proof of the spectral resolution theorem in the form in which it was originally proved by von Neumann: for it shows that the hypotheses of Theorem I follow from those of von Neumann.

THEOREM II. *If H is a self-adjoint operator, then for any ϕ in the domain of H there exists a function $\psi(t)$ for all t satisfying*

$$\psi(0) = \phi, \qquad \frac{1}{i}\frac{d\psi}{dt} = H\psi. \tag{18}$$

Consider the operator

$$R_z = (H-z)^{-1}.$$

If $y \neq 0$, this is a bounded operator.†

Let $$w(z;f,f) = w(z) = (R_z f, f).$$

Then $w(z)$ is defined for every f in R for all points not on the real axis in the z-plane.

Then,‡ if $y > \eta > 0$, $w(z) = u+iv$,

$$w(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(\lambda+i\eta)}{\lambda+i\eta-z}\,d\lambda,$$

and $v(\lambda+i\eta)$ is positive and $L(-\infty,\infty)$ in λ.

Consider the integral

$$\int_{-X}^{X} w(x+iy)e^{i(x+iy)t}\,dx = \frac{1}{\pi}\int_{-X+iy}^{X+iy} e^{izt}\,dz \int_{-\infty}^{\infty} \frac{v(\lambda+i\eta)}{\lambda+i\eta-z}\,d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} v(\lambda+i\eta)\,d\lambda \int_{-X+iy}^{X+iy} \frac{e^{izt}}{\lambda+i\eta-z}\,dz.$$

Since the inner integral tends boundedly to its limit, we get, for $t < 0$,

$$\int_{iy-\infty}^{iy+\infty} w(z)e^{izt}\,dz = 2i\int_{-\infty}^{\infty} v(\lambda+i\eta)e^{i(\lambda+i\eta)t}\,d\lambda.$$

Put $$p_1(\phi,\phi;t) = \frac{1}{2\pi i}\int_{i\alpha-\infty}^{i\alpha+\infty} w(z;\phi,\phi)e^{izt}\,dz \quad (\alpha > 0). \tag{19}$$

† See Stone, Definition 4.1 and Theorem 4.18.

‡ See Lengyel, *Acta Szeged*, 9 (1939), 174–86. For a more general proof see Titchmarsh, *Quart. J. of Math.* (Oxford), 12 (1941), 89–107.

Then, since†

$$\int_{-\infty}^{\infty} v(\lambda+i\eta)\,d\lambda = \pi\|\phi\|^2,$$

we see that

$$p_1(\phi,\phi;t) = \begin{cases} 0 & (t>0), \\ \frac{1}{2}\|\phi\|^2 & (t=0), \end{cases}$$

$$p_1(\phi,\phi;t) \leqslant \|\phi\|^2 \qquad (t<0).$$

We can now infer the existence of a functional

$$p_1(\phi,g;t) = \frac{1}{2\pi i}\int_{i\alpha-\infty}^{i\alpha+\infty} (R_z\phi,g)e^{izt}\,dz \quad (\alpha>0),$$

for, plainly,

$$4p_1(\phi,g;t) = p_1(\phi+g;\phi+g;t) - p_1(\phi-g;\phi-g;t) + \\ +i[p_1(\phi+ig,\phi+ig;t) - p_1(\phi-ig,\phi-ig;t)].$$

It is easy to see that

$$|p_1(\phi,g;t)| \leqslant (\|\phi\|^2+\|g\|^2)$$

and, from this, that

$$|p_1(\phi,g,t)| \leqslant 2\|\phi\|\,\|g\|. \tag{20}$$

Further, the function

$$p_2(\phi,g;t) = -\frac{1}{2\pi i}\int_{-i\alpha-\infty}^{i\alpha+\infty} (R_z\phi,g)e^{izt}\,dt$$

exists and has properties similar to those of p_1, save that p_2 vanishes for $t>0$ instead of $t<0$. The functional

$$p(\phi,g;t) = p_1(\phi,g;t)+p_2(\phi,g;t)$$

exists for all t and satisfies (20). Since p is linear in both ϕ and g, it is easy to infer from (20) that, for any ϕ, p is a bounded functional of g, and hence there exists a point $\psi(t)$ in Hilbert space for any t such that for any g‡

$$\{\psi(t),g\} = p(\phi,g;t).$$

We can now prove that $\psi(t)$ satisfies (18). In the first place, it follows readily from

$$p(\phi,\phi;0) = \|\phi\|^2$$

that

$$p(\phi,g,0) = (\phi,g),$$

and hence that

$$\psi(0) = \phi.$$

† See Lengyel, loc. cit.

‡ See Stone, Theorem 2, 27.

Secondly, we have $$R_z = \frac{HR_z - I}{z}.$$

Let C be a contour running from negative to positive infinity in the upper half-plane, and in the reverse direction in the lower half-plane. Then, if ϕ and g are in D,

$$\begin{aligned} p(\phi, g; t) &= \frac{1}{2\pi i} \int_C (R_z \phi, g) e^{izt}\, dz \\ &= \frac{1}{2\pi i} \int_C (HR_z \phi, g) \frac{e^{izt} - 1}{z}\, dz + \frac{1}{2\pi i} \int_C (HR_z \phi, g) \frac{dz}{z} - \\ &\qquad - \frac{1}{2\pi i} \int_C (\phi, g) e^{izt} \frac{dz}{z} \\ &= i \int_0^t p(H\phi, g; \tau)\, d\tau + \text{constant}, \end{aligned}$$

since the third integral is equal to (ϕ, g), and the second is convergent.

We have, therefore, for any g in R

$$p(H\phi, g; t) = \frac{1}{i} \frac{d}{dt} p(\phi, g; t).$$

Since H and R_z are commutable operators, it is easy to show from the definitions that

$$p(H\phi, g; t) = p(\phi, Hg; t),$$

so that we have $$\frac{1}{i} \frac{d}{dt} p(\phi, g; t) = p(\phi, Hg; t),$$

i.e. $$\frac{1}{i} \frac{d}{dt} \{\psi(t), g\} = \{\psi(t), Hg\} = \{H\psi(t), g\},$$

provided that $\psi(t)$ is in D. But, for every g in D,

$$\begin{aligned} |\{\psi(t), Hg\}| &= |p(H\phi, g; t)| \\ &\leqslant 2\|H\phi\|\, \|g\|, \end{aligned}$$

so that $\{\psi(t), Hg\}$ is a bounded linear functional in g provided that ϕ is in D. Hence $\psi(t)$ is in the domain of H^*, which is equal to H, since H is assumed self-adjoint. Hence the differential equation (18) is satisfied.

The converse of Theorem II can be proved by an argument which is substantially a reversal of that above; but the following proof is

of greater interest, as it indicates what must happen in those cases when there is not a solution of (1) for all t.

THEOREM III. *If H is a symmetric operator such that the equations* (1) *have a solution in D for all elements in the domain D of H, then H is essentially self-adjoint.*

If ϕ is any element in D and $\psi(t)$ the corresponding solution of (1), we know from equation (2) that

$$\|\psi(t)\| = \|\phi\| = \text{constant},$$

and if ϕ_1 and ϕ_2 are any two elements in D, we can prove, in precisely the same way as (2) was proved, that

$$\{\psi_1(t), \psi_2(t)\} = (\phi_1, \phi_2). \tag{21}$$

It follows that the transformation which takes ϕ into $\psi(t)$ is isometric. Denote this transformation by $U(t)$. The transformation $U(-t)$ is plainly the inverse transformation of $U(t)$, and by hypothesis both $U(t)$ and $U(-t)$ have the whole of the Hilbert space R as their domains. It therefore follows that $U(t)$ is unitary for all t.†

If $\{\phi_n\}$ is a complete normal orthogonal set of elements in D, the elements $\psi_n(t)$, the corresponding solutions of (1) into which they are transformed by $U(t)$, also form a complete normal orthogonal set in R, lying in D.

After these preliminaries we can prove that H is essentially self-adjoint. For this it is sufficient to show that, if f, g are any two elements in the domain D^* of H^*, the adjoint of H, then

$$(H^*f, g) = (f, H^*g), \tag{22}$$

for this proves that H^* is adjoint to itself, and so that $H^* \subseteq H^{**}$. Since $H^* \supseteq H$, we have $H^{**} \subseteq H^*$, so that $H^* = H^{**}$ and H is essentially self-adjoint.‡

For f and g in D^*,

$$\begin{aligned}(H^*f, g) &= \sum_1^\infty \{H^*f, \psi_m(t)\}\{\psi_m(t), g\} \qquad (23)\\ &= \sum_1^\infty \{f, H\psi_m(t)\}\{\psi_m(t), g\}\\ &= \sum_1^\infty i\left(f, \frac{d\psi_m}{dt}\right)\{\psi_m(t), g\}.\end{aligned}$$

† Stone, Def. 2.18.

‡ Stone, 43, and Def. 2.12.

In the same way,

$$(f, H^*g) = \sum_1^\infty \frac{1}{i}\{f, \psi_m(t)\}\left(\frac{d\psi_m}{dt}, g\right),$$

and hence

$$(H^*f, g) - (f, H^*g) = i \sum \frac{d}{dt}[\{f, \psi_m(t)\}\{\psi_m(t), g\}]. \tag{24}$$

This last series will now be shown to be boundedly convergent. To do this it is sufficient to show that the series on the right hand of (23) is boundedly convergent, since the series (24) is the sum of (23) and a similar, conjugate, series. Let $M_N(t)$ be the closed linear manifold determined by the elements $\psi_n(t)$ $(n = 1, 2, \ldots, N)$, and let $E_N(t)$ be the corresponding projection operator. Then

$$\left|\sum_1^N \{H^*f, \psi_n(t)\}\{\psi_n(t), g\}\right| = |\{E_N(t)H^*f, E_N(t)g\}|$$
$$\leqslant \|H^*f\|\,\|g\|,$$

for all N and t, so that the series is boundedly convergent. Because of this we are justified in integrating (24) term by term, and we get from (24)

$$(H^*f, g) - (f, H^*g) = i\frac{d}{dt}\sum \{g, \psi_m(t)\}\{\psi_m(t), g\}$$
$$= i\frac{d}{dt}\|g\|^2 = 0.$$

This proves the theorem.

4. It is instructive to consider the situation when (1) does not have a solution for all t. There are a number of possibilities: for example, (1) may have a solution for all elements in R for all positive t but not for all negative t, or vice versa; or it may have solutions for some elements but not for others; or it may not have solutions for any elements at all. Examples of these cases are given by the transformation which is defined to be equal to $\frac{1}{i}\frac{df}{dx}$ for all functions $f(x)$ which are:

(a) in the space $L^2(0, \infty)$ together with their differential coefficients, and satisfy $f(0) = 0$;

(b) in the space $L^2(0, 1)$ with their differential coefficients, and satisfy $f(0) = f(1) = 0$;

(c) in the space $L^2(0, 1)$ with their differential coefficients, satisfy $f(0) = f(1) = 0$, and are analytic.

It is easily verified that all these operators are symmetric. The solution of (1) corresponding to $\phi = f(x)$ is $\psi(t) = f(x+t)$, provided that the latter is an element of the domain of H. In the first case no solution exists for any positive t save for functions with $f(x) = 0$ for $x \leqslant t$, and there is no element save the nul element for which a solution exists for all positive t. For $t < 0$ the solution is $\psi(t) = f(x+t)$, where $f(x) = 0$ for $x \leqslant 0$. In the second case no solution exists for either positive or negative t except for functions with $f(x) = 0$ for $0 \leqslant x \leqslant t$ if t is positive, or with $1+t \leqslant x \leqslant 1$ if t is negative, and for no element save the nul element is there a solution either for all positive or for all negative t. In the third case no solution exists for any element save the nul element for any t whether positive or negative.

It is quite easy to give examples where solutions exist for, say, all positive t, for some but not all elements of D.

Let us now consider the situation when (1) has a solution for all positive t but not for all negative t, in the light of the proof of Theorem III. In that case the transformation $U(t)$ can be defined as before, and it is an isometric transformation;† but the range of $U(t)$ cannot determine the closed linear manifold R, for, if it did, $U(t)$ would be unitary and would possess an inverse, in which case the function

$$\psi(t) = U(-t)^{-1}\phi \quad (t \text{ negative})$$

would be a solution of (1) for all negative t. Thus $U(t)$ transforms the space R into a proper sub-space, $R(t)$ say. Let the corresponding projection operator be $E(t)$. Then, as in the proof of Theorem III,

$$(H^*g, g) - (g, H^*g) = i\frac{d}{dt}\sum \{g, \psi_m(t)\}\{\psi_m(t), g\},$$

but now this sum is equal to $\|E(t)g\|^2$ and so, for $t = 0$,

$$2I(H^*g, g) = \frac{d}{dt}\|E(t)g\|^2.$$

Since $R(t)$ is a proper sub-set of R for any positive t, there must exist g for which

$$\frac{d}{dt}\|E(t)g\|^2 < 0, \quad \text{for } t = 0,$$

and for these $I(H^*g, g) < 0$, so that g is in the C^- class of H.‡

† Stone, Def. 2.19. ‡ Stone, 344.

In the same way, if there is a solution of (1) for all negative but not all positive t, the C^+ class of H is not empty. If there are elements for which no solution exists for both positive and negative t, a similar argument proves the existence of both C^+ and C^- classes.

These statements are easily verified in the case of the transformations cited above. Thus the transformation $U(t)$ corresponding to the example (a) exists for negative values of t and transforms the space $L^2(0,\infty)$ into the sub-space $L^2(-t,\infty)$. $E(t)g$ is the function which is zero for $0 \leqslant x < -t$, and equal to $g(x)$ for $x \geqslant -t$. For $t = 0$,

$$\frac{d}{dt}\|E(t)g\|^2 = \frac{d}{dt}\int_{-t}^{\infty} \|g(x)\|^2\,dx = \|g(0)\|^2$$

and this is equal to $2I(H^*g, g)$.

In conclusion, I wish to express my thanks to Professor Titchmarsh for comments on the manuscript of this paper, to the referee for his criticisms, and in particular for the present formulation of condition (a) of Theorem I, and to Professor Tyndall for permission to use the Physics Library of the University of Bristol.

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